# LINEAR ALGEBRA: LECTURES 19–20

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We started with the following puzzle. Suppose there are four turtles (Alice, Bob, Carol, and Dick) standing at the four corners of a unit square. They simultaneously begin crawling, with Alice always crawling directly towards Bob, Bob crawling directly towards Carol, Carol crawling directly towards Dick, and Dick crawling directly towards Alice:
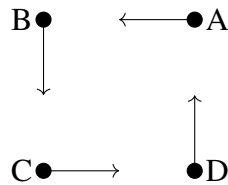

As they crawl, the four turtles spiral in towards the center of the original square, and will eventually meet. How far has Alice crawled in total (from start until all four turtles meet)? There's a way to do this in your head without any messy calculation! We'll return to this later in the lecture.

Next, we formalized something we've already been doing implicitly. In general, points and vectors are very different types of objects, and should be treated as such. However, linear maps are blind to the distinction between points and vectors. Since in this course we are studying linear maps, we will henceforth abuse notation and use the same notation for vectors as we do for points. For example, we would write the vector $a\vec{e}_1 + b\vec{e}_2$ as $\begin{pmatrix} a \\ b \end{pmatrix}$. If we were dealing with non-linear maps, this would be a really bad idea, but so long as we only work with linear maps we're safe.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and has positive determinant. Last time we proved that f can be written as the composition of three simple maps:

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta.$$

We also explored what the image of a square grid looks like. The canonical square grid will generally get mapped to some strange shear grid:
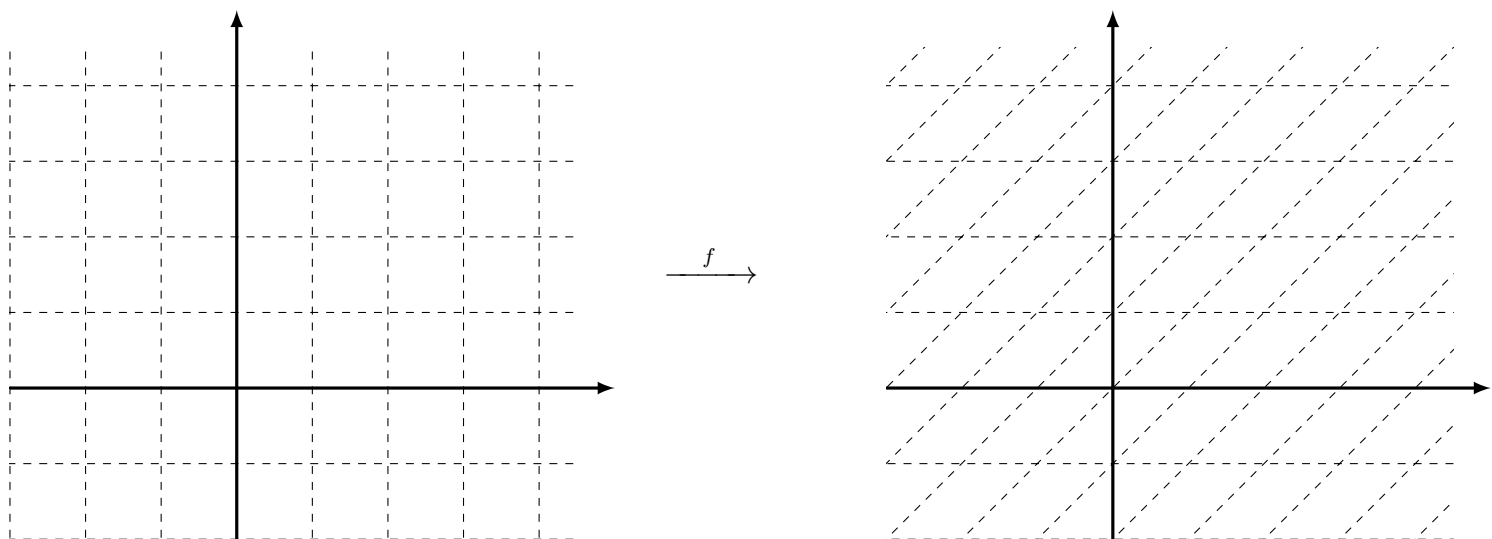
However, it's desirable to find a square grid which gets mapped to a rectangular grid, like this:
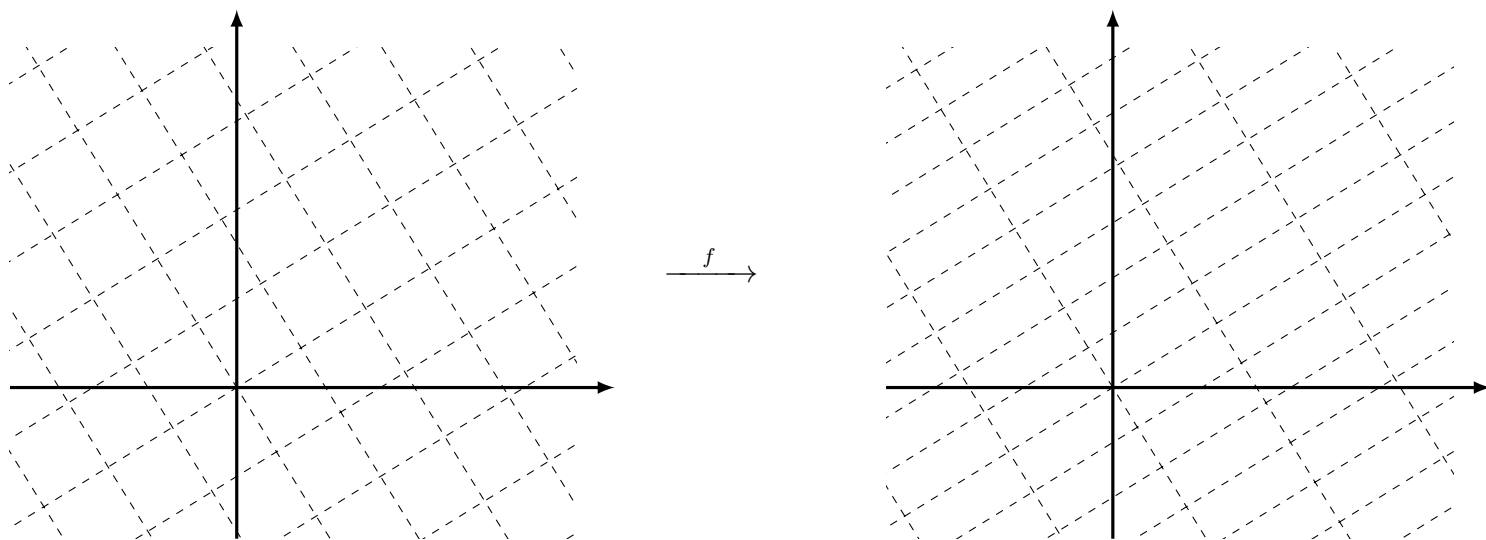
Why are perpendicular grids better than other ones? Because motions along perpendicular directions are totally independent of one another. For example, if you walk east, there's no component of north in your walk. By contrast, there is a component of northeast in your walk. This insight allows us to resolve the turtle problem from the beginning of the lecture. Note that by symmetry, the turtles will always be located at the corners of a square. It follows that the direction of Alice's and Bob's motions are always perpendicular. Thus, *Bob's motion is neither taking him closer nor further from Alice.* So far as Alice is concerned, Bob might as well be standing still as she crawls towards him. We conclude that Alice crawls a total of one unit until she meets up with Bob (and, by symmetry, the rest of the gang).

What does any of this have to do with the singular value decomposition? Recall that the image $f(U)$ of the unit circle U is an ellipse. It follows that its major and minor radii, which we think of as vectors and label $\vec{v}$ and $\vec{w}$, are perpendicular to one another. Let $\alpha := arg\vec{v}$, i.e., the angle between $\vec{v}$ and $\vec{e}_1$. Let $k := |\vec{v}|$ and $\ell := |\vec{w}|$. Here's a picture:
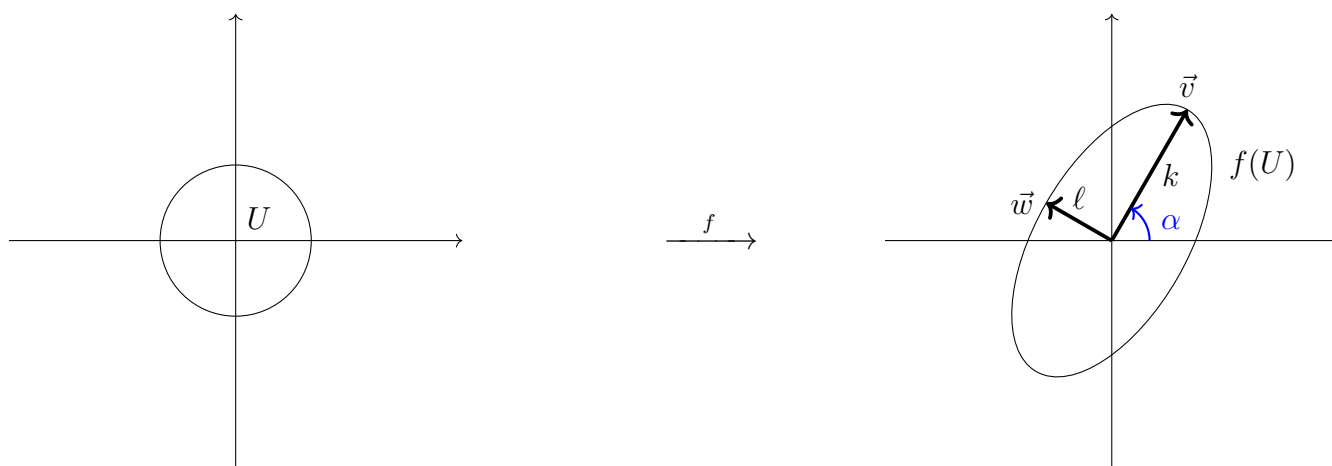
From our work last time, we know that the singular value decomposition of f is

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta$$

where β is some angle. What does β represent in the above picture?

Proposition 1. *Given $f : \mathbb{R}^2 \to \mathbb{R}^2$ linear with positive determinant, and suppose it has the SVD*

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta$$

Let $\vec{v}$ denote the radius in the first quadrant of the ellipse $f(U)$, and $\vec{w}$ the radius in the second quadrant. Then

$$f^{-1}(\vec{v}) = R_{-\beta}(\vec{e}_1) \qquad and \qquad f^{-1}(\vec{w}) = R_{-\beta}(\vec{e}_2).$$

Proof. We have

$$f\left(R_{-\beta}\vec{e}_1\right) = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta R_{-\beta}\vec{e}_1$$

$$= R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} \vec{e}_1$$

$$= R_\alpha(k\vec{e}_1)$$

$$= \vec{v}.$$

(Make sure you can justify all the above equalities!) Since f has positive determinant, it is nonsingular, hence invertible. This shows that $f^{-1}(\vec{v}) = R_{-\beta}(\vec{e}_1)$ as claimed. The second claim can be proved similarly. $\qquad\square$
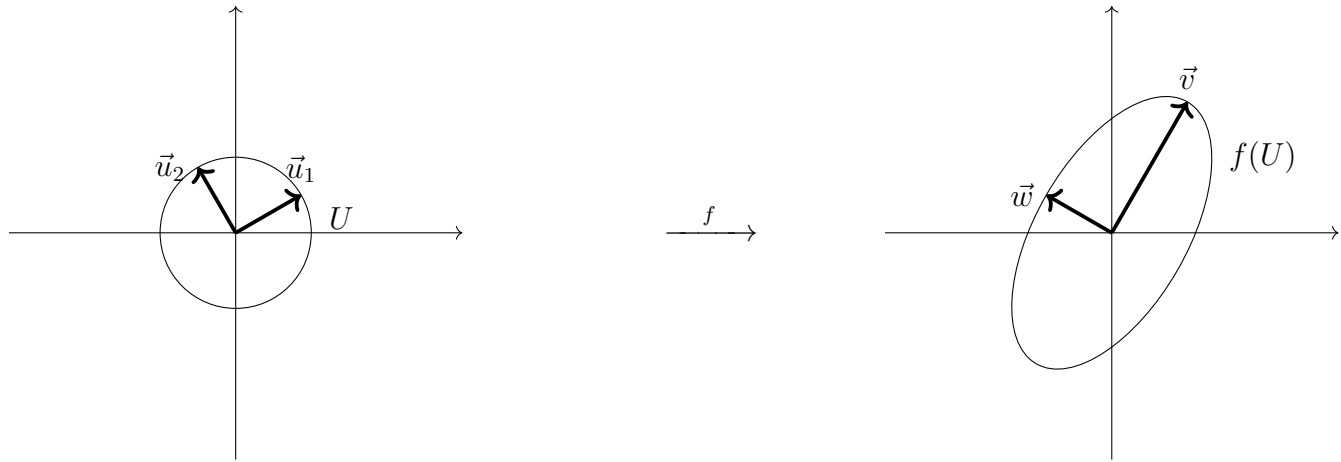
We immediately deduce the following.

Corollary 2. *Let f, $\vec{v}$, and $\vec{w}$ be as above. Then $f^{-1}(v) \perp f^{-1}(\vec{w})$.*

Make sure you can prove this!

Finally, we can prove that there exists a rectangular grid which gets mapped by f to another rectangular grid. Set

$$\vec{u}_1 := f^{-1}(\vec{v}) \qquad and \qquad \vec{u}_2 := f^{-1}(\vec{w}).$$

In pictures:

By our corollary above, we know that $\vec{u}_1 \perp \vec{u}_2$. It follows that the grid

$$S := \{m\vec{u}_1 + n\vec{u}_2 : m, n \in \mathbb{Z}\}$$

is rectangular. (In fact, since both $\vec{u}_i$'s are unit vectors, S is a square grid.) It's straightforward to verify that

$$f(S) = \{m\vec{v} + n\vec{w} : m, n \in \mathbb{Z}\}$$

which is also rectangular since $\vec{v} \perp \vec{w}$.

This is all well and good, but how does one actually go about determining the Singular Value Decomposition of a given map f?

STEP 1. Determine $\vec{v}$ and $\vec{w}$.

> To do this, consider an arbitrary point $(\cos\theta, \sin\theta) \in U$. Its image is the point $f(\cos\theta, \sin\theta)$ on the ellipse $f(U)$. Which θ gives the radii $\vec{v}$ and $\vec{w}$? Those which maximize / minimize the magnitude $|f(\cos\theta, \sin\theta)|$. Thus we have a calculus problem: find the values of θ which maximize and minimize $|f(\cos\theta, \sin\theta)|$, and then plug them in to find $\vec{v}$ and $\vec{w}$. [*One suggestion: rather than maximizing $|f(\cos\theta, \sin\theta)|$, maximize $|f(\cos\theta, \sin\theta)|^2$. It's much easier.*]

STEP 2. Determine the magnitudes $k := |\vec{v}|$ and $\ell := |\vec{w}|$, as well as $\alpha := \arg \vec{v}$. Let $\vec{u}_1 := f^{-1}\vec{v}$, and set $\beta := -\arg \vec{u}_1$. [*Note that you've already determined $\vec{u}_1$ – and hence β – in the previous step.*]

STEP 3. WIN! We have the SVD of f:

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta.$$

You will practice this process on the next homework.

We ended the lecture by computing the first few powers

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$$

We noticed that the resulting matrices were composed of Fibonacci numbers. We will discuss this further next lecture.