LINEAR ALGEBRA: LECTURE 21

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We started by making explicit something we've dealt with before: function composition is *associative*. What does this mean? Given three function $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$, consider the composition $f \circ g \circ h$. This is clearly a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, but there is something fishy about this expression. The operation $\circ$ is a way of combining two functions to form a single function; we're trying to compose *three* functions, but we can only compose two at a time. The question is, does $f \circ g \circ h$ mean $(f \circ g) \circ h$ or $f \circ (g \circ h)$? A bit of thought shows that either is fine:

$$\Big((f \circ g) \circ h\Big)(x) = (f \circ g)\Big(h(x)\Big) = f\Big(g\Big(h(x)\Big)\Big) = f\Big((g \circ h)(x)\Big) = \Big(f \circ (g \circ h)\Big)(x)$$

for any x, whence $(f \circ g) \circ h = f \circ (g \circ h)$. To sum up, even though the symbol $f \circ g \circ h$ is in principle ambiguous, it is perfectly well-defined – the two reasonable interpretations of the symbol turn out to be the same. This property of $\circ$ is called associativity. (By contrast, the operation $\div$ is *not* associative: $4 \div 2 \div 2$ either equals 1 or 4, depending on your interpretation of where the parentheses go.) One consequence of $\circ$'s associativity is that

$$f^2 \circ f = f \circ f^2,$$

since both are interpretations of $f \circ f \circ f$.

Having discussed associativity, we turned to Fibonacci numbers. Recall the Fibonacci sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

where each term is the sum of the two previous terms and the sequence begins with two 1's. For convenience, let us label the n^{th} term of this sequence by f_n (so that, for example, $f_4 = 3$). It is also convenient to define $f_0 := 0$. We can give a more succinct definition of the Fibonacci sequence via the recursion

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+1} = f_n + f_{n-1} \quad \forall n \geq 1.$$

Question. *What's f_{1000}? More generally, is there an efficient way to calculate f_n for any n?*

Last time we empirically discovered a connection between Fibonacci numbers and powers of a certain matrix. We state and prove this now.

Proposition 1. *For all integers $n \geq 1$ we have* $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$.

Proof. We prove this by a method known as *induction*. A quick computation shows that the claim holds for $n = 1$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^1 = \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix}$$

Next, suppose that we happen to know that for some particular integer n,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}.$$

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Then (by associativity) we have

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{n+1} = \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{n}\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}f_{n+1} & f_n\\f_n & f_{n-1}\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$$
$$= \begin{pmatrix}f_{n+1}+f_n & f_{n+1}\\f_n+f_{n-1} & f_n\end{pmatrix} = \begin{pmatrix}f_{n+2} & f_{n+1}\\f_{n+1} & f_n\end{pmatrix}.$$

Thus, we have proved that

whenever the claim holds true for some integer n, it must continue to hold true for the integer $n+1$.

At the very beginning of this proof we saw that the claim holds for $n = 1$. Applying the italicized sentence above, we immediately deduce (without any calculation!) that the claim must also hold for $n = 2$. Again applying the italicized sentence we find that the claim must also hold for $n = 3$. We can continue in this way as long as we like, therefore showing that the claim holds for all integers larger than 1.[1] □

A proof of this type is called a proof by *induction* because it involves showing that if some property holds for the number n, then this induces the property for the number $n+1$.

The above result suggests an approach to finding the 1000th Fibonacci number: evaluate $\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{999}$. At first glance, this seems more difficult than the original question of determining the 1000th Fibonacci number – composing the map $\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$ with itself 999 times doesn't sound appealing. However, we will develop a method which will allow us to raise this matrix to a large power very easily. To do this, we first discuss the concept of *similarity*.

Given a line $\mathcal{L}$ through the origin, let $\sigma_{\mathcal{L}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection across $\mathcal{L}$; on your midterm, you proved that $\sigma_{\mathcal{L}}$ is linear. How could one determine the matrix of $\sigma_{\mathcal{L}}$? It's far from obvious by brute force. However, we can change our point of view to greatly simplify our desired operation: first rotate $\mathcal{L}$ until it's horizontal, then apply $\rho = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, and finally rotate $\mathcal{L}$ back to its original position. In symbols:

$$\sigma_{\mathcal{L}} = R_{-\theta}\rho R_\theta$$

where θ is chosen so that $R_\theta(\mathcal{L})$ is horizontal. We know the matrices of all three of the linear maps on the right hand side, which means it is now a straightforward calculation to determine the matrix of $\sigma_{\mathcal{L}}$.

Here's another example of this trick. We know how to rotate around the origin by θ; simply apply the map R_θ. What if we wish to instead rotate by θ around some other point $p \in \mathbb{R}^2$? First translate the plane until p is in the position of the origin; then apply the standard rotation map; then translate back to put p in its original location. In symbols: $T_p R_\theta T_{-p}$.

In both examples above, we performed a complicated operation by first changing our perspective, then performing a similar (but simpler!) operation, then changing our perspective back to the original point of view. This motivates the following definition.

Definition. Given two linear maps $f, g : \mathbb{R}^2 \to \mathbb{R}^2$. We say f is *similar* to g, denoted $f \sim g$, if and only if there exists a nonsingular linear map P such that $f = P^{-1}gP$.

The intuition is as above: to understand f, we first put on some glasses which change our perception of the plane (i.e., we apply P), then perform some simpler-to-understand operation (i.e., apply the map g), then remove our glasses to look upon our work. (The choice of the word *similar* is because often f and g are similar types of functions. For example, $\sigma_{\mathcal{L}} \sim \rho$, and both are reflections across a line.)

Proposition 2. *Similarity enjoys the following properties:*

(1) (Reflexivity) $f \sim f$

[1] If you find this unsatisfying, perhaps you would prefer this proof by contradiction. Suppose the claim is false at some point, and let N denote the smallest integer at which the claim fails. Then the claim must hold at $N-1$. But we just showed that whenever the claim holds at an integer, it must continue to hold at the next largest integer. Contradiction!

(2) (Symmetry) *If $f \sim g$, then $g \sim f$*
(3) (Transitivity) *If $f \sim g$ and $g \sim h$, then $f \sim h$*

Any relation which satisfies these three properties is called an *equivalence relation*; thus $\sim$ is an equivalence relation. Another example of an equivalence relation is the notion of congruence $\cong$ from Euclidean geometry. However, equivalence relations aren't restricted to math. All stereotyping is based on an implicit equivalence relation. For example, we could consider two people equivalent iff they have the same hair color; this effectively reduces any person to their hair color. This method of comparing two people is another example of an equivalence relation. (Can you verify this?)

What does any of this have to do with our original question about Fibonacci numbers? Well, suppose $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \sim g$ for some map g. By definition, this means

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = P^{-1}gP.$$

This implies that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = P^{-1}gPP^{-1}gP = P^{-1}g^2P.$$

Similarly, for any $n \geq 1$ we have

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = P^{-1}g^nP.$$

Thus, if we can find a matrix g which is similar to $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and is easy to raise to powers, we'll be able to make progress on our Fibonacci problem. There are many examples of maps which are easy to raise to powers. One set of particularly nice ones are the diagonal matrices:

$$\begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix}^n = \begin{pmatrix} k^n & 0 \\ 0 & \ell^n \end{pmatrix}.$$

Thus our new goal becomes to determine a diagonal map $g \sim \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. We will take this up next lecture.