LINEAR ALGEBRA: LECTURE 22–24

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1. A Comment on Notation

Thus far, we've been writing points in the form (a, b) or $\begin{pmatrix} a \\ b \end{pmatrix}$ and vectors in the form $a\vec{e}_1 + b\vec{e}_2$. I've been insisting on this to emphasize the distinction between points and vectors. However, as we've seen, linear maps can't distinguish between the two. Since this is a course on linear algebra (and therefore concerned almost exclusively with linear maps) we will use point notation to denote both points and vectors. In other words, we will write $\begin{pmatrix} a \\ b \end{pmatrix}$ instead of $a\vec{e}_1 + b\vec{e}_2$.

2. An Explicit Formula for Fibonacci Numbers

Recall that we are trying to find a formula for the n^{th} Fibonacci number f_n using matrices. Our strategy (described last time) is to find a diagonal matrix which is similar to $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, say,

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = P^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} P. \tag{1}$$

We thus rephrase our goal as follows:

Question 1. *Do there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ and an invertible matrix P such that (1) holds?*

If the answer to this question is affirmative, then it would follows that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} P = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

whence

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} P(\vec{e}_1) = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\vec{e}_1) = P(\lambda_1 \vec{e}_1) = \lambda_1 P(\vec{e}_1). \tag{2}$$

Further, note that $P(\vec{e}_1) \neq \mathbf{0}$, since we are hoping to find an invertible map P. (Can you justify this sentence?) Thus, to have any hope of answering Question 1 in the affirmative, we must be able to find a number λ and a vector $\vec{v}$ such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v} = \lambda \vec{v}. \tag{3}$$

In other words, we wish to find some nonzero vector $\vec{v}$ such that when we apply $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ to it we get the same vector back, just stretched out by a factor of λ. Note that no matter what λ and $\vec{v}$ are, there's one map which has the desired effect:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vec{v} = \lambda \vec{v}.$$

Thus, we wish to find λ and $\vec{v}$ such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vec{v}$$

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or equivalently

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \vec{v} = \mathbf{0}.$$

Recall from above that $\vec{v} \neq \mathbf{0}$. What sort of linear map sends a nonzero vector to the zero vector? Only a singular one! (Can you explain why?) Thus, we deduce that we must have

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0,$$

or in other words,

$$\lambda^2 - \lambda - 1 = 0.$$

Recall that we're searching for a number λ and a nonzero vector $\vec{v}$ which satisfy the relation (3). What we've just proved is that if these exist, then

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

Set

$$\lambda_1 := \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 := \frac{1-\sqrt{5}}{2}$$

Returning to (3), it remains to find some nonzero vector $\vec{v}_1$ such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v}_1 = \lambda_1 \vec{v}_1.$$

How do we construct such a vector? The most natural approach is to write $\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$, plug it in above, and solve for x and y. When we did this in class, we discovered that $x = \lambda_1 y$ and that y satisfies $(\lambda_1^2 - \lambda_1 - 1)y = 0$. Note that the latter relationship holds for every y! (Why is this?) Although this looks like a failure at first glance, it's actually a success – this tells us that we can choose y to be anything, and then set $x = \lambda_1 y$. For example, we can take

$$\vec{v}_1 := \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

With this choice, it is easy to verify that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v}_1 = \lambda_1 \vec{v}_1.$$

Going back to (2) shows that we'd like to find an invertible map P such that

$$P(\vec{e}_1) = \vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

The exact same arguments show that we'd like

$$P(\vec{e}_2) = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

Combining the two previous statements tells us how to choose P:

$$P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

Now that we've determined λ_1, λ_2, and P, it's straightforward to verify (1).

Having done all of this, it's not so hard to find an explicit formula for the n^{th} Fibonacci number. Manipulating (1), we see that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$$

(make sure you can explain why!). Raising both sides to the n^{th} power gives

$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} & * \end{pmatrix}$$

It follows that

$$f_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

where λ_1, λ_2 are the two solutions to the equation $\lambda^2 - \lambda - 1 = 0$. We've discovered our formula!

3. Spectral Theory

The key to figuring out the formula for f_n above was finding numbers λ_1 and λ_2, along with an invertible matrix P, such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$$

In addition to being useful for calculating powers of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, this gives us a nice geometric interpretation of the action of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ on the plane. Recall from above that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v}_j = \lambda_j \vec{v}_j$$

where $P(\vec{e}_1) = \vec{v}_1$ and $P(\vec{e}_2) = \vec{v}_2$. In other words, P is the change-of-basis map from $\vec{e}_1, \vec{e}_2$ to $\vec{v}_1, \vec{v}_2$, and if we replace the usual coordinate system (generated by $\vec{e}_1$ and $\vec{e}_2$) by the one generated by $\vec{v}_1$ and $\vec{v}_2$, then $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has a simple geometric description: it stretches the plane out by a factor of λ_1 in the $\vec{v}_1$ direction and by λ_2 in the $\vec{v}_2$ direction.

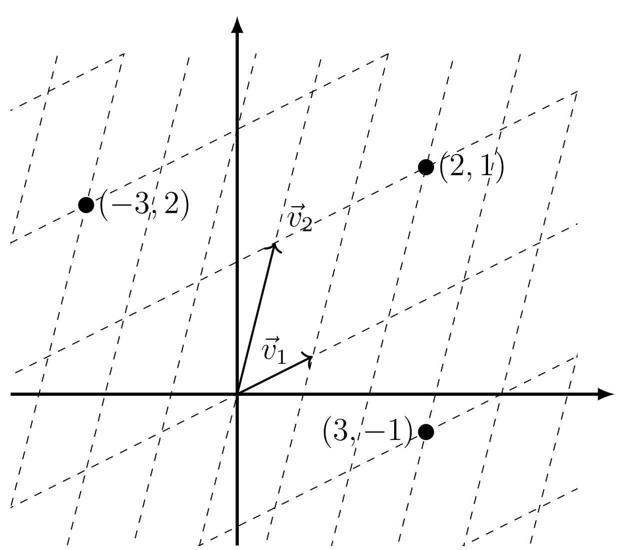
Let's generalize this. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map. Suppose we can find numbers λ_1, λ_2 and an invertible matrix P such that

$$f = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}.$$

This is called the *spectral decomposition of* f, and it gives us a nice way to interpret f. Think of P as the change-of-basis matrix from $\vec{e}_1, \vec{e}_2$ to some vectors $\vec{v}_1, \vec{v}_2$. Then it is straightforward to verify that

$$f(\vec{v}_1) = \lambda_1 \vec{v}_1 \qquad \text{and} \qquad f(\vec{v}_2) = \lambda_2 \vec{v}_2$$

Now label each point of the plane in terms of how to get there using $\vec{v}_1$ and $\vec{v}_2$. For example:



Once we adopt this perspective, it's very easy to describe what f does to any point: it stretches the first coordinate by λ_1 and the second by λ_2. For example, where does f send the point $(3, -2)$ indicated above? (Note: this isn't the usual $(3, -2)$; it's $3\vec{v}_1 - 2\vec{v}_2$.) Easy: $f(3, -2) = (3\lambda_1, -2\lambda_2)$. The quantities λ_j and $\vec{v}_j$ play a pivotal role in understanding the spectral decomposition, so they get a special name.

Definition. Given a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$. We say a number λ is an *eigenvalue* of f if and only if there exists a nonzero vector $\vec{v}$ such that

$$f(\vec{v}) = \lambda\vec{v}.$$

In this case, we say $\vec{v}$ is an *eigenvector* corresponding to the eigenvalue λ.

Example 1. We discovered above that the map $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

and that their corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

One immediate remark is that eigenvectors aren't uniquely determined.

Proposition 1. *Suppose f has eigenvalue λ with corresponding eigenvector $\vec{v}$. Then $\alpha\vec{v}$ is an eigenvector corresponding to λ for every $\alpha \neq 0$.*

3.1. **Finding the spectral decomposition.** We now generalize the process used to analyze $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ to determine the spectral decomposition of an arbitrary linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$. We break the process into a few steps.

<u>STEP 1.</u> Solve the equation $\det(f - \lambda I) = 0$ for λ, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix. The eigenvalues of f are the solutions to this equation.

> *Why does this work? Suppose λ is an eigenvalue of f. Then by definition, there exists a nonzero $\vec{v}$ such that $f(\vec{v}) = \lambda\vec{v}$. Just as in the Fibonacci example, this happens iff $\exists\vec{v} \neq \mathbf{0}$ such that $(f - \lambda I)\vec{v} = \mathbf{0}$. But this is occurs iff the map $f - \lambda I$ is singular.*

<u>STEP 2.</u> Solve the equation $f(x, 1) = (\lambda x, \lambda)$ for x. Then $\vec{v} := \begin{pmatrix} x \\ 1 \end{pmatrix}$ is an eigenvector corresponding to λ.

> *Why does this work? We found an eigenvalue λ above, and we wish to find a corresponding eigenvector $\vec{v}$. Since any rescaling of $\vec{v}$ remains an eigenvector, we may as well rescale in such a way that $\vec{v} = \begin{pmatrix} x \\ 1 \end{pmatrix}$. Now by definition, $\vec{v}$ must satisfy the equation $f(\vec{v}) = \lambda\vec{v}$.*

<u>STEP 3.</u> Say the two eigenvalues of f are λ_1 and λ_2, with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} b \\ d \end{pmatrix}$.

Let $P := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $f = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$.

> *Why is this? A good exercise!*

Before exploring specific examples, let's try to predict what sorts of problems might arise in the steps above. In Step 1, we might only find a single eigenvalue; when this happens, this usually bodes ill for the spectral decomposition, as we shall see below. In Step 2, it's possible that $\vec{e}_1$ is an eigenvector, in which case no renormalization would make it possess the form we described. Finally, in Step 3, we need to worry about the possibility that P is not invertible. These fears are all justified, and some of these problems are fatal to the process. In fact, as we shall see, *not every linear map admits a spectral decomposition*. By contrast, every linear map admits a Singular Value Decomposition. (On the other hand, when a map *does* admit a spectral decomposition, it's much easier to find than the SVD.)

Let's explore a few representative examples.

Ex. 1. $f = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}$

To find the eigenvalues, we first solve the equation

$$\det(f - \lambda I) = 0$$

for λ. The LHS is

$$\det\left(\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = \det \begin{pmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 15.$$

Expanding this, setting equal to zero, and solving yields $\lambda = -2, 6$. Let's set

$$\lambda_1 := -2 \qquad \text{and} \qquad \lambda_2 := 6.$$

These are the eigenvalues.

Next, we find corresponding eigenvectors. We first solve the equation

$$\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = -2 \begin{pmatrix} x \\ 1 \end{pmatrix}$$

From this we easily deduce that $x = -1$, whence our first eigenvector is $\vec{v}_1 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Similarly, solving

$$\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = 6 \begin{pmatrix} x \\ 1 \end{pmatrix}$$

yields $x = 3/5$, whence $\vec{v}_2 := \begin{pmatrix} 3/5 \\ 1 \end{pmatrix}$. If we wish, we can make this look nicer by rescaling it to $\vec{v}_2 := \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

The final step of the process is to determine the change of basis map P:

$$P := \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$$

Thus, our spectral decomposition is

$$\begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} = P \begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix} P^{-1}$$

Ex. 2. $R_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

As before, we begin by finding the eigenvalues of $R_{\pi/2}$ via the equation

$$\det(R_{\pi/2} - \lambda I) = 0.$$

This equation can be rewritten as

$$\lambda^2 + 1 = 0,$$

so the eigenvalues of $R_{\pi/2}$ are $\lambda_1 = i$ and $\lambda_2 = -i$. Next, we find the corresponding eigenvectors. Write $\vec{v}_1 = \begin{pmatrix} x \\ 1 \end{pmatrix}$; we're supposed to solve

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = i \begin{pmatrix} x \\ 1 \end{pmatrix},$$

which immediately yields $x = i$. It follows that $\vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$. A similar argument shows that $\vec{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. Finally, let P be the change-of-basis

$$P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}.$$

Then we have the spectral decomposition

$$R_{\pi/2} = P \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} P^{-1}.$$

Note that we've successfully found a spectral decomposition of $R_{\pi/2}$, but only if we allow ourselves to use imaginary numbers. This is a bit odd, since the original function $R_{\pi/2}$ has nothing to do with imaginary numbers! This hints at a connection between rotations in $\mathbb{R}^2$ and complex numbers. On the other hand, a bit more thought shows that it's not unreasonable that the spectral decomposition of a rotation should be unusual, since a rotation doesn't stretch the plane in any direction.

<u>Ex. 3.</u> $g = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

Following the above procedure, we find that the only eigenvalue of g is $\lambda = 2$. Continuing along shows that the only eigenvectors of g are scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It follows that any change-of-basis matrix P would not be invertible, which means that g has no spectral decomposition. An alternative way to express this is that g is *not diagonalizable*.

Note that we could have seen that g wasn't diagonalizable without solving for the eigenvalues. For, suppose g did have a spectral decomposition. Since we know the only eigenvalue is 2, we would be able to write

$$g = P \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$$

for some matrix P. But this would immediately imply $g = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, which isn't the case! Put differently, we've just shown that the only diagonalizable matrix with both eigenvalues equal to 2 is $2I$.