LINEAR ALGEBRA: LECTURE 31

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Recall that a linear map between two vector spaces V and W is a function $T : V \to W$ which is additive and scales.[1] A particularly nice type of linear map is the following:

Definition. A linear map $T : V \to W$ is called *invertible* iff $\#T^{-1}(\vec{w}) = 1$ for any $\vec{w} \in W$. Here $T^{-1}(\vec{w})$ is the *pre-image* of $\vec{w}$:

$$T^{-1}(\vec{w}) := \{\vec{v} \in V : T(\vec{v}) = \vec{w}\}.$$

If T is invertible, we call T^{-1} the *inverse* of T.

Colloquial version. A map is invertible iff every vector in the target space comes from precisely one vector in the source space.

Remark. This is different from the definition given in the textbook.

Intuitively, if T is invertible then T^{-1} and T should undo one another. The following makes this precise:

Lemma 1. *Suppose $T : V \to W$ is an invertible linear map. Then*

(i) $T^{-1} \circ T(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$

(ii) $T \circ T^{-1}(\vec{w}) = \vec{w}$ for all $\vec{w} \in W$

Proof. We first prove (ii). Pick any $\vec{w} \in W$, and note that $T^{-1}\vec{w} \neq \emptyset$. By definition, $T\vec{v} = \vec{w}$ for every $\vec{v} \in T^{-1}\vec{w}$. Claim (ii) immediately follows.

Next we turn to (i). Pick $\vec{v} \in V$. By definition,

$$\begin{aligned} T^{-1} \circ T(\vec{v}) &= T^{-1}\Big(T\vec{v}\Big) \\ &= \{\vec{a} \in V : T\vec{a} = T\vec{v}\} \end{aligned}$$

whence $\vec{v} \in T^{-1}\Big(T\vec{v}\Big)$. Since T is invertible, $\vec{v}$ must be the *only* element of $T^{-1}\Big(T\vec{v}\Big)$, and (i) follows. □

Given an invertible map $T : V \to W$, we call the function $T^{-1} : W \to V$ the inverse of T. This function has a number of nice properties. For example:

Proposition 2. *If $T : V \to W$ is invertible, then $T^{-1} : W \to V$ is linear.*

Proof. We need to show that T^{-1} is additive and scales. We'll show the former, and leave the latter as an exercise.

Given $\vec{w}_1, \vec{w}_2 \in W$. Since T is invertible, there exist unique $\vec{v}_1, \vec{v}_2 \in V$ such that $T\vec{v}_i = \vec{w}_i$ for $i = 1, 2$. Applying the lemma above yields

$$T^{-1}(\vec{w}_1 + \vec{w}_2) = T^{-1}\Big(T\vec{v}_1 + T\vec{v}_2\Big) = T^{-1}\Big(T(\vec{v}_1 + \vec{v}_2)\Big) = \vec{v}_1 + \vec{v}_2 = T^{-1}\vec{w}_1 + T^{-1}\vec{w}_2$$

which shows that T^{-1} is additive. □

Next we played the 'game of 15'. After a couple of rounds, we realized that this game is *isomorphic* to tic-tac-toe: they're the same game, but played with different symbols. In particular, each move in one game corresponds to a unique move in the other game, and a sequence of moves leads to a win (or a draw) in one game if and only if the corresponding sequence of moves in the other game leads to the same conclusion.

The above is an isomorphism between games. It turns out there's a simple notion of isomorphism between vector spaces:

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[1] In the text this is called a *linear transformation*.

Definition. A vector space V is *isomorphic* to a vector space W, denoted $V \simeq W$, if and only if there exists a linear invertible map $T : V \to W$. Such a T is called an *isomorphism.*

Colloquial version. Two vector spaces are isomorphic iff one of them is just the other one with its elements relabelled. The isomorphism mapping one to the other is a dictionary which translates the names. One might reasonably hope that there's a reverse dictionary as well. This turns out to be the case:

Proposition 3. *If $T : V \to W$ is an isomorphism, then $T^{-1} : W \to V$ is also an isomorphism.*

Proof. We must show that T^{-1} is invertible, i.e., that given any $\vec{v} \in V$ there exists a unique $\vec{w} \in W$ such that $T^{-1}\vec{w} = \vec{v}$.

Existence. By our Lemma above, $T^{-1}\Big(T\vec{v}\Big) = \vec{v}$.

Uniqueness. Suppose $T^{-1}\vec{v}_1 = \vec{w} = T^{-1}\vec{v}_2$. Then (again by our Lemma above) we would have

$$\vec{v}_1 = T \circ T^{-1}\vec{v}_1 = T \circ T^{-1}\vec{v}_2 = \vec{v}_2$$

□