LINEAR ALGEBRA: LECTURE 32

LEO GOLDMAKHER

Last time we discussed isomorphisms between vector spaces. Most pairs of vector spaces are not isomorphic, so linear maps between them are usually not invertible. We can quantify the extent to which a linear map fails to be invertible using the concepts of *image* and *kernel*:

Definition. Given a linear map $T : V \to W$.

- The *image of* T is
$$\text{im } T := \{T\vec{v} : \vec{v} \in V\} = T(V).$$
- The *kernel of* T is
$$\ker T := \{\vec{v} \in V : T\vec{v} = \vec{0}\}$$

Colloquial version. The image of T is the set of all outputs of T; the kernel of T is the set of all vectors annihilated by T. Roughly speaking, the image tells you how broadly T broadcasts information, while the kernel tells you how much T compresses information.

Remark. The textbook calls the image the 'range' of T.

Example 1. Consider $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) := (x, y)$. What are the image and kernel of this map?

Claim. im $f = \mathbb{R}^2$.

Proof. Pick any $(a, b) \in \mathbb{R}^2$. Since $(a, b) = f(a, b, 0)$, we see that $(a, b) \in$ im f. Thus $\mathbb{R}^2 \subseteq$ im f. On the other hand, im $f \subseteq \mathbb{R}^2$. The claim follows. □

Claim. ker $f = \{(0, 0, z) : z \in \mathbb{R}\}$.

Proof. It is obvious that $\{(0, 0, z) : z \in \mathbb{R}\} \subseteq$ ker f, so it remains to check that no other types of points live in the kernel. Suppose $(a, b, c) \in$ ker f. Then $(0, 0) = f(a, b, c) = (a, b)$, whence $(a, b, c) = (0, 0, c)$. It follows that ker $f \subseteq \{(0, 0, z) : z \in \mathbb{R}\}$. The claim follows. □

Example 2. Consider $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $g(x, y, z) := (x - y, y - z, x - z)$. What are the image and kernel of this map?

Claim. im $g = \{(a, b, a + b) : a, b \in \mathbb{R}\}$.

Proof. As above, we'll prove this by showing that each set is a subset of the other. First, pick any element $p \in$ im g; by definition of g, $p = (x - y, y - z, x - z)$ for some $x, y, z \in \mathbb{R}$. It follows that $p \in \{(a, b, a + b) : a, b \in \mathbb{R}\}$. In the other direction, consider any point of the form $(a, b, a + b)$. Since $(a, b, a + b) = g(a, 0, -b) \in$ im g, we see that $\{(a, b, a + b) : a, b \in \mathbb{R}\} \subseteq$ im g. The claim is proved. □

Claim. ker $g = \{(a, a, a) : a \in \mathbb{R}\}$.

Proof. Once again we prove this by showing each is a subset of the other. It's obvious that the RHS is a subset of the LHS. Now suppose $(a, b, c) \in$ ker g. Then $(0, 0, 0) = g(a, b, c) = (a - b, b - c, a - c)$, whence $a = b = c$. Thus, every element of ker g is of the form (a, a, a). The claim follows. □

In the above examples, the images and outputs have nice structure. In the first example, the image is the plane $\mathbb{R}^2$, while the kernel is a line. (In fact, it's isomorphic to $\mathbb{R}$. Can you see why?) In the second example, the image is a plane (isomorphic to $\mathbb{R}^2$) while the kernel is a line (isomorphic to $\mathbb{R}$). In both situations, we see that the image and the kernel are vector spaces in their own right. This inspires the following notion:

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Definition. Given a vector space V. A *subspace* of V is any subset $A \subseteq V$ which is a vector space with respect to the same operations as V.

Proposition 1. *Suppose $T : V \to W$ is a linear map. Then* im T *is a subspace of W, and* ker T *is a subspace of V.*

Proof. We illustrate a couple of properties, and leave the rest as exercises. Let's check that im T is closed under addition. Pick any $\vec{w}_1, \vec{w}_2 \in \text{im}\, T$; we wish to show that $\vec{w}_1 + \vec{w}_2 \in \text{im}\, T$. By definition, $\exists \vec{v}_i \in V$ such that $T\vec{v}_i = \vec{w}_i$ for $i = 1, 2$. It follows that

$$\vec{w}_1 + \vec{w}_2 = T\vec{v}_1 + T\vec{v}_2 = T(\vec{v}_1 + \vec{v}_2) \in \text{im}\, T,$$

so we see that im T is closed under addition.

Next, let's verify that addition in im T is commutative. Pick any $\vec{w}_1, \vec{w}_2 \in \text{im}\, T$. Then in particular, $\vec{w}_1, \vec{w}_2 \in W$. Since W is a vector space, $\vec{w}_1 + \vec{w}_2 = \vec{w}_2 + \vec{w}_1$, which proves commutativity in im T. A short way to express this is: *commutativity in* im T *is inherited from the ambient vector space W.*

We leave the rest of the proof as an exercise. □

Observe that in both of our examples we found that image to have dimension 2 and the kernel to have dimension 1. This is not always the case, of course. However, the fact that these dimensions sum to the dimension of the source space $\mathbb{R}^3$ is not an accident!

Theorem 2 (Rank-Nullity theorem)**.** *Given any linear map $T : V \to W$. Then*

$$\dim(\text{im}\, T) + \dim(\text{ker}\, T) = \dim V.$$

Colloquial version. This says that the total amount of data transmitted by T combined with the total amount of data compression yields the total amount of data prior to transmission.

We will discuss this theorem (as well as prove it) next time.