LINEAR ALGEBRA: LECTURE 33

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We begain by proving the theorem we finished with last lecture.

Theorem 1 (Rank-Nullity theorem)**.** *Given any linear map $T : V \to W$ where V and W are finite dimensional vector spaces. Then*

$$\dim(\text{im } T) + \dim(\text{ker } T) = \dim V.$$

Colloquial version. This says that the total amount of data transmitted by T combined with the total amount of data compression yields the total amount of data prior to transmission.

Proof. Since ker T is a subspace of V, it must also be a finite-dimensional vector space, hence admits a basis $\mathcal{B}_{\text{ker}} := \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$. In particular, this set of vectors is linearly independent, so (by problem **8.1**) it is contained in a basis of V, say,

$$\mathcal{B}_V = \{\vec{u}_1, \ldots, \vec{u}_k, \vec{v}_1, \ldots, \vec{v}_i\}.$$

Finally, set

$$\mathcal{B}_{\text{im}} := \{T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_i\}.$$

I claim that $\mathcal{B}_{\text{im}}$ is a basis of im T. We prove this in the usual way, using the Fundamental Property of Bases.

Spanning. Given $\vec{w} \in$ im T, we wish the show that $\vec{w}$ can be written as a linear combination of the elements in $\mathcal{B}_{\text{im}}$. Since $\vec{w} \in$ im T, there exists $\vec{v} \in V$ such that $T\vec{v} = \vec{w}$. Write $\vec{v}$ in terms of the basis $\mathcal{B}_V$, say,

$$\vec{v} = \alpha_1\vec{u}_1 + \cdots + \alpha_k\vec{u}_k + \beta_1\vec{v}_1 + \cdots + \beta_i\vec{v}_i.$$

Then we have

$$\begin{aligned} \vec{w} = T\vec{v} &= T(\alpha_1\vec{u}_1) + \cdots + T(\alpha_k\vec{u}_k) + T(\beta_1\vec{v}_1) + \cdots + T(\beta_i\vec{v}_i) \\ &= \beta_1 T\vec{v}_1 + \cdots + \beta_i T\vec{v}_i. \end{aligned}$$

This shows that $\mathcal{B}_{\text{im}}$ spans im T. //

Linear Independence. Suppose

$$\gamma_1 T\vec{v}_1 + \cdots + \gamma_i T\vec{v}_i = \vec{0}.$$

Then $T(\gamma_1\vec{v}_1 + \cdots + \gamma_i\vec{v}_i) = \vec{0}$, whence $\gamma_1\vec{v}_1 + \cdots + \gamma_i\vec{v}_i \in$ ker T. Write this vector in terms of our basis $\mathcal{B}_{\text{ker}}$, say,

$$\gamma_1\vec{v}_1 + \cdots + \gamma_i\vec{v}_i = \delta_1\vec{u}_1 + \cdots + \delta_k\vec{u}_k.$$

It follows that

$$\gamma_1\vec{v}_1 + \cdots + \gamma_i\vec{v}_i - \delta_1\vec{u}_1 - \cdots - \delta_k\vec{u}_k = \vec{0}.$$

However, the set of vectors appearing in this linear combination are linearly independent (since they form the basis $\mathcal{B}_V$). It follows that all the coefficients must be 0. In particular,

$$\gamma_1 = \gamma_2 = \cdots = \gamma_i = 0.$$

This shows that the only linear combination of the elements of $\mathcal{B}_{\text{im}}$ which produces $\vec{0}$ is the trivial one. //

We've shown that $\mathcal{B}_{\text{im}}$ is a basis of im T. It follows that $\dim(\text{im } T) = i$. We also know from above that $\dim(\text{ker } T) = k$ and $\dim V = k + i$. This concludes the proof of the rank-nullity theorem. $\square$

Date: May 6, 2016.

One nice way to think about dimension is as a measure of the size of a vector space. For example, which is larger: the space of 3×3 magic squares, or R^2? Both of these have infinitely many elements, so it's hard to say. However, they have different dimension: the dimension of magic square space is 3, while the dimension of $\mathbb{R}^2$ is 2. Thus the space of 3×3 magic squares is 'larger' than the plane $\mathbb{R}^2$.

We reinterpret the Rank-Nullity theorem from this perspective. Fix vector spaces V and W. The theorem asserts that the smaller the kernel of T is, the more faithfully the image of T captures V in terms of the elements of W. In particular, the smaller the kernel of T is, the closer T is to being an isomorphism. We will make this precise below. But first, we demonstrate the utility of the Rank-Nullity theorem.

Proposition 2. *Suppose V and W are finite-dimensional vector spaces. Then V is isomorphic to W if and only if $\dim V = \dim W$.*

Proof. We prove the two statements separately.

($\Rightarrow$)

If V is isomorphic W, then there exists an invertible linear map $T : V \to W$. In particular, we see that $\ker T = \{\vec{0}\}$, and also that $T^{-1}(\vec{w}) \neq \emptyset$ for every $\vec{w} \in W$. From the latter we deduce that $W = \text{im } T$. The Rank-Nullity theorem implies

$$\dim V = \dim W$$

as claimed. //

($\Leftarrow$)

Given $\dim V = \dim W$. Then there exists a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of V and a basis $\{\vec{w}_1, \ldots, \vec{w}_n\}$ of W. Define a map $T : V \to W$ by setting

$$T(\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n) := \alpha_1 \vec{w}_1 + \cdots + \alpha_n \vec{w}_n$$

for any scalars $\alpha_i \in \mathbb{R}$. It's straightforward to check that T is an invertible linear map. //

This concludes the proof. $\square$