LINEAR ALGEBRA: LECTURE 34

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In the first half of this course, we dealt with linear transformations of the plane; we found that it was convenient to represent these maps in the form of a 2×2 matrix (although it's worth pointing out that many of our proofs were matrix-free). More recently, we've been studying linear maps between arbitrary vector spaces. The goal of today's lecture is to describe any such linear map as a matrix. As you shall see, this process is highly reminiscent of the two-dimensional case.

We first discuss *coordinates* in a vector space. Given a finite-dimensional vector space V, pick a basis of V, say, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Then (by definition) any vector $\vec{v} \in V$ can be expressed as a linear combination of the $\vec{v}_i$'s in a unique way: there exists a unique $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n.$$

We abuse notation and write

$$\vec{v} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

This notation explains why we call it a vector space – we can express any element of the vector space using coordinates, just like we're used to with ordinary vectors.

A word of caution. Coordinates only make sense once you have specified a basis (and an order on the elements of the basis). As we have seen, a given vector space has many different bases, and the same vector might have totally different coordinates depending on which basis you pick.

Recall that we represented linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ by a 2×2 matrix, whose first column was where $\vec{e}_1$ got sent and second column where $\vec{e}_2$ got sent. We now develop an analogous matrix representation for abstract linear maps. Suppose $T : V \to W$ is a linear map between two finite-dimensional vector spaces. Pick a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of V and a basis $\{\vec{w}_1, \ldots, \vec{w}_m\}$ of W. Then we write the matrix of T as

$$T = \begin{pmatrix} T\vec{v}_1 & T\vec{v}_2 & \cdots & T\vec{v}_n \end{pmatrix}$$

where each $T\vec{v}_i$ is a column of numbers. What numbers? Well, since $T\vec{v}_1 \in W$, we can write $T\vec{v}_1$ in terms of the basis $\{\vec{w}_1, \ldots, \vec{w}_m\}$, say,

$$T\vec{v}_1 = \alpha_1 \vec{w}_1 + \cdots + \alpha_m \vec{w}_m = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$$

This is the first column of our matrix for T. We do the same for all the remaining columns. Let's work out a couple of examples.

Example 1. Recall that $\mathbb{P}_n := \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : a_i \in \mathbb{R}\ \forall i\}$ (in words: $\mathbb{P}_n$ is the set of all polynomials of degree at most n). Let's try to write down the matrix of the differential operator $\frac{d}{dx} : \mathbb{P}_3 \to \mathbb{P}_2$.

<u>STEP 1</u>. Identify bases of the source and target spaces.

As we've discussed previously, a natural basis for $\mathbb{P}_n$ is $\{x^0, x^1, x^2, \ldots, x^n\}$. We use this both for $\mathbb{P}_2$ and $\mathbb{P}_3$.

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STEP 2. Evaluate where the map sends the basis elements.

We have

$$\frac{d}{dx}x^0 = 0 = 0x^0 + 0x^1 + 0x^2 = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Similarly, we find

$$\frac{d}{dx}x^1 = 1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \frac{d}{dx}x^2 = 2x = \begin{pmatrix}0\\2\\0\end{pmatrix} \qquad \frac{d}{dx}x^3 = 3x^2 = \begin{pmatrix}0\\0\\3\end{pmatrix}$$

STEP 3. Write down the matrix.

Making the above images our columns, we find

$$\frac{d}{dx} = \begin{pmatrix}0 & 1 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 3\end{pmatrix}$$

Let's verify that this makes sense, by differentiating a polynomial using this matrix:

$$\frac{d}{dx}(2x^3 - 9x^2 + 5) = \begin{pmatrix}0 & 1 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 3\end{pmatrix}\begin{pmatrix}5\\0\\-9\\2\end{pmatrix}$$

Evaluating this in the usual way – taking the dot product of each row of the matrix by the column vector – we find

$$\begin{pmatrix}0 & 1 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 3\end{pmatrix}\begin{pmatrix}5\\0\\-9\\2\end{pmatrix} = \begin{pmatrix}0\\-18\\6\end{pmatrix}$$

Translating these coordinates into an actual polynomial gives $6x^2 - 18x$. It works!

Example 2. Recall that $MSS_3(\mathbb{R})$ denotes the vector space of all 3×3 magic squares with real entries. Let $W := \{(x, y, z) \in \mathbb{R}^3 : z = x + y\}$, and consider the linear map $T : MSS_3(\mathbb{R}) \to W$ defined by

$$T\left(\begin{array}{|c|c|c|}\hline a & b & * \\\hline * & * & * \\\hline * & * & * \\\hline\end{array}\right) := (a, b, a + b).$$

What is the matrix of T?

STEP 1. Identify bases of the source and target spaces.

Recall from the solution to problem **7.4(c)** that a basis of $MSS_3(\mathbb{R})$ is given by $\{\vec{m}_1, \vec{m}_2, \vec{m}_3\}$ where

$$\vec{m}_1 := \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\\hline 1 & 1 & 1 \\\hline 1 & 1 & 1 \\\hline\end{array} \qquad \vec{m}_2 := \begin{array}{|c|c|c|}\hline 1 & -1 & 0 \\\hline -1 & 0 & 1 \\\hline 0 & 1 & -1 \\\hline\end{array} \qquad \vec{m}_3 := \begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\\hline 1 & 0 & -1 \\\hline -1 & 1 & 0 \\\hline\end{array}$$

A nice basis of W is $\{\vec{w}_1, \vec{w}_2\}$ where

$$\vec{w}_1 := (1, 0, 1) \qquad \text{and} \qquad \vec{w}_2 := (0, 1, 1).$$

STEP 2. Evaluate where the map sends the basis elements.

We have

$$T\vec{m}_1 = (1,1,2) = \vec{w}_1 + \vec{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly, we find

$$T\vec{m}_2 = \vec{w}_1 - \vec{w}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad T\vec{m}_3 = -\vec{w}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

STEP 3. Write down the matrix.

Making the above images our columns, we find

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

Having illustrated the concept with two examples, we return to our abstract discussion. First, note that a given linear map $T : V \to W$ can be represented by infinitely many different matrices! Indeed, the matrix of T depends very strongly on the bases we pick for V and W. *One of the overarching goals of linear algebra is, given a linear map $T : V \to W$, to find bases of V and W which make the matrix of T as simple as possible.* For example, given a linear map $T : V \to V$, the matrix of T is a square (it will be $n \times n$, where $\dim V = n$); the simplest form a square matrix can take is a diagonal matrix. Can one find a basis of V such that the corresponding matrix of T is diagonal? The answer is: sometimes, but not always. We will return to this topic on Friday.