Instructor: Leo Goldmakher

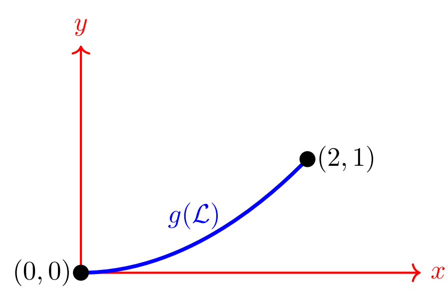
Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Midterm Exam 1 – KEY

M1–1 Consider $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$g(x, y) := (x, y^2),$$
and let $\mathcal{L}$ be the line segment connecting $(0, 0)$ to $(2, 1)$. What is the image $g(\mathcal{L})$? Sketch a picture, and give as precise a mathematical description as you can.

First observe that we can express $\mathcal{L}$ as a set of points:
$$\mathcal{L} = \{(2t, t) : 0 \le t \le 1\}.$$
It follows that
$$g(\mathcal{L}) = \{g(2t, t) : 0 \le t \le 1\} = \{(2t, t^2) : 0 \le t \le 1\}.$$
If we label the first coordinate as x and the second as y, we see that all of these points satisfy the condition $y = \frac{1}{4}x^2$. Thus, the points of $g(\mathcal{L})$ are precisely those points on the parabola $y = \frac{1}{4}x^2$ with $0 \le x \le 2$. Here's a picture:



M1–2 Carefully explain why $f\big(f^{-1}(x)\big) = x$ for any $x \in \text{im } (f)$. What happens if $x \notin \text{im } (f)$?

Recall that the preimage of x is defined to be
$$f^{-1}(x) := \{y : f(y) = x\}.$$
If $x \notin \text{im } (f)$, then $f\big(f^{-1}(x)\big) = f(\emptyset) = \emptyset$. If $x \in \text{im } (f)$, then $f^{-1}(x) \ne \emptyset$, whence
$$\begin{aligned} f\big(f^{-1}(x)\big) &= f\big(\{y : f(y) = x\}\big) \\ &= \big\{f(y) : f(y) = x\big\} \\ &= \{x\} \\ &= x \end{aligned}$$
(recall our convention that sets consisting of a single element are indistinguishable from that element).

M1–3 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. In class we showed that the image of the unit square whose lower left vertex is at the origin has area $\det f$. Prove that this is true for an arbitrary unit square in the plane.

We first prove the special case of the 'upright' square:

Lemma 1. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, and let S denote the unit square whose lower left corner is the origin. If S' is a translation of S, then the area of $f(S')$ is* $\det f$.

Proof. Label the lower left corner of S' as (p, q), so $S' = (p, q) + S$. Then the other corners of S' are located at $(p + 1, q)$, $(p, q + 1)$, and $(p + 1, q + 1)$. Problem **4.7** implies that the image of S' is the quadrilateral with vertices

$$f(p, q), \quad f(p + 1, q), \quad f(p + 1, q + 1), \quad \text{and} \quad f(p, q + 1).$$

By additivity, we can rewrite these in the form

$$f(p, q) + f(0, 0), \quad f(p, q) + f(1, 0), \quad f(p, q) + f(1, 1), \quad \text{and} \quad f(p, q) + f(0, 1).$$

Thus $f(S') = f(p, q) + f(S)$, i.e., $f(S')$ is simply a translation of $f(S)$. It follows that

$$\text{area } f(S') = \text{area } f(S) = \det f$$

as claimed. □

Our goal is to bootstrap from this special case to the general case. But first, we arm ourselves with one more preparatory result:

Lemma 2. *Given an arbitrary unit square $T \subseteq \mathbb{R}^2$, there exists an angle α such that $R_\alpha(T)$ is an upright square (i.e., the sides of $R_\alpha(T)$ are parallel to the coordinate axes).*

Proof. If T is upright, we're done (take $\alpha = 0$). Thus we may suppose T is not upright. It follows that T has a side with positive slope; label the endpoints of this side A and B. We define a function $g : \mathbb{R} \to \mathbb{R}$ by setting $g(\theta)$ to be the slope of the line segment $R_\theta(\overline{AB})$. Note that g is a continuous function, and that $g(0) > 0 > g\left(\frac{\pi}{2}\right)$. By the intermediate value theorem, there exists an α between 0 and $\frac{\pi}{2}$ such that $g(\alpha) = 0$. Thus one side of $R_\alpha(T)$ is parallel to the horizontal axis; this implies $R_\alpha(T)$ is an upright square. □

We are now in a position to handle the general case.

Proposition 3. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. If T is an arbitrary unit square in the plane, then* $\text{area } f(T) = \det f$.

Proof. By Lemma 2, there exists $\alpha \in \mathbb{R}$ such that $S' := R_\alpha(T)$ is an upright unit square. It follows that $T = R_{-\alpha}(S')$, whence

$$f(T) = (f \circ R_{-\alpha})(S').$$

Lemma 1 implies that

$$\begin{aligned} \text{area } f(T) &= \det(f \circ R_{-\alpha}) \\ &= (\det f)(\det R_{-\alpha}) \\ &= \det f \end{aligned}$$

since for any θ we have $\det R_\theta = \cos^2\theta + \sin^2\theta = 1$. □

M1–4 In class we've considered several times the linear map $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects across the horizontal axis. In this problem we explore the more general reflection $\sigma_{\mathcal{L}} : \mathbb{R}^2 \to \mathbb{R}^2$ across a given line $\mathcal{L}$.

(a) Prove that $R_\theta \circ \rho = \rho \circ R_{-\theta}$.

From class we know that

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{and} \qquad \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus

$$R_\theta \circ \rho = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

while

$$\rho \circ R_{-\theta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

The claim follows. $\square$

(b) Prove that if $\mathcal{L}$ is a line passing through the origin, then $\sigma_{\mathcal{L}}$ is linear. [*Hint: Write $\sigma_{\mathcal{L}}$ as a composition of linear maps.*]

[See illustration on next page.] To find where the reflection map $\sigma_{\mathcal{L}}$ sends a point x, we can tilt our head until $\mathcal{L}$ is horizontal (or vertical), then flip over this axis, and then finally untilt our head. Let's translate this into the language of linear algebra:

$$\sigma_{\mathcal{L}} = R_\theta \circ \rho \circ R_{-\theta}$$

(For a rigorous proof, see Lemma below.) Since all three of these operations are linear, their composition must also be linear. $\square$

Lemma 4. $\sigma_{\mathcal{L}} = R_\theta \circ \rho \circ R_{-\theta}$

Proof due to Yuxin Wu. Let θ denote the angle formed between $\mathcal{L}$ and the positive horizontal axis. Set

$$p := R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad q := R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\sigma_{\mathcal{L}}(p) = p$ and $\sigma_{\mathcal{L}}(q) = -q$. It follows that

$$\sigma_{\mathcal{L}} \circ R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_{\mathcal{L}} \circ R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = R_\theta \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

whence

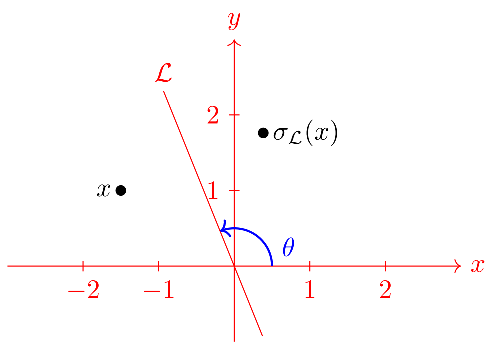
$$R_{-\theta} \circ \sigma_{\mathcal{L}} \circ R_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad R_{-\theta} \circ \sigma_{\mathcal{L}} \circ R_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

This implies that

$$R_{-\theta} \circ \sigma_{\mathcal{L}} \circ R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho.$$

The claim follows. $\square$

Label as θ the angle formed by $\mathcal{L}$ and the positive part of the horizontal axis:



An illustration of $\sigma_{\mathcal{L}}$

(c) Suppose $\mathcal{L}$ and $\mathcal{L}'$ are two distinct lines in the plane, both passing through the origin. Describe $\sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}$ geometrically, with justification. [*Hint: Use parts (a) and (b).*]

Applying part (a) to part (b), we see that $\sigma_{\mathcal{L}} = R_\alpha \circ \rho$ for some α. (In fact, $\alpha = 2\theta$, where θ is the angle illustrated above.) Similarly, $\sigma_{\mathcal{L}'} = R_\beta \circ \rho$ for some β. Applying (a) yields

$$\sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'} = R_\alpha \circ \rho \circ R_\beta \circ \rho = R_\alpha \circ R_{-\beta} \circ \rho \circ \rho = R_{\alpha-\beta}.$$

Thus $\sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}$ is a rotation.

M1–5 We say a function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is *distance-preserving* iff

$$|\phi(x) - \phi(y)| = |x - y| \qquad \forall x, y \in \mathbb{R}^2.$$

In other words, the distance between the images of any two points is the same as the distance between the two points themselves.

(a) Give an example of a distance-preserving function which is not a linear map.

Let $T_k : \mathbb{R}^2 \to \mathbb{R}^2$ be the translation map, defined by

$$T_k(x) := x + k.$$

Then for any two points $x, y \in \mathbb{R}^2$, we have

$$|T_k(x) - T_k(y)| = |(x + k) - (y + k)| = |x - y|$$

so T_k is distance-preserving. On the other hand, T_k cannot be linear for any $k \neq \mathbf{0}$, since $T_k(\mathbf{0}) \neq \mathbf{0}$.

(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is distance-preserving and satisfies $f(\mathbf{0}) = \mathbf{0}$. Prove that $|f(x)| = |x|$ for all $x \in \mathbb{R}^2$.

By the definition of distance-preserving, we have

$$|f(x)| = |f(x) - f(\mathbf{0})| = |x - \mathbf{0}| = |x|$$

for any $x \in \mathbb{R}^2$. $\square$

(c) Suppose f is as in (b). Prove that $f(x) \cdot f(y) = x \cdot y$ for all $x, y \in \mathbb{R}^2$. [*Hint: Start with the distance-preserving relation* $|f(x) - f(y)| = |x - y|$.]

Using properties of dot products, as well as part (b), we see

$$\begin{aligned} |f(x) - f(y)|^2 &= \big(f(x) - f(y)\big) \cdot \big(f(x) - f(y)\big) \\ &= f(x) \cdot f(x) - 2\big(f(x) \cdot f(y)\big) + f(y) \cdot f(y) \\ &= |f(x)|^2 + |f(y)|^2 - 2\big(f(x) \cdot f(y)\big) \\ &= |x|^2 + |y|^2 - 2\big(f(x) \cdot f(y)\big). \end{aligned}$$

On the other hand, since f is distance preserving, we have

$$|f(x) - f(y)|^2 = |x - y|^2 = (x - y) \cdot (x - y) = x \cdot x - 2(x \cdot y) + y \cdot y = |x|^2 + |y|^2 - 2(x \cdot y).$$

The claim immediately follows. $\square$

(d) Suppose f is as in (b). Prove that f must be linear. [*Hint: First prove that* $|f(\alpha x) - \alpha f(x)| = 0$.]

As usual, to prove that f is linear we must verify that it's additive and scales.
Scaling. Given $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$ we have

$$\begin{aligned} |f(\alpha x) - \alpha f(x)|^2 &= \big(f(\alpha x) - \alpha f(x)\big) \cdot \big(f(\alpha x) - \alpha f(x)\big) \\ &= f(\alpha x) \cdot f(\alpha x) - 2\alpha\big(f(x) \cdot f(\alpha x)\big) + \alpha^2\big(f(x) \cdot f(x)\big) \\ &= (\alpha x) \cdot (\alpha x) - 2\alpha(x \cdot \alpha x) + \alpha^2(x \cdot x) \\ &= 0. \end{aligned}$$

It follows that $|f(\alpha x) - \alpha f(x)| = 0$, whence $f(\alpha x) = \alpha f(x)$ as claimed.

Additivity. Given $x, y \in \mathbb{R}^2$, we have

$$|f(x + y) - f(x) - f(y)|^2 = \big(f(x + y) - f(x) - f(y)\big) \cdot \big(f(x + y) - f(x) - f(y)\big).$$

Expanding this, using part (b), and simplifying yields

$$|f(x + y) - f(x) - f(y)|^2 = 0.$$

This implies that $f(x + y) = f(x) + f(y)$ as claimed.

Combining both of the above properties proves that f must be linear. $\square$

(e) Suppose f is as in (b). Prove that there exists $\theta \in \mathbb{R}$ such that either $f = R_\theta$ or $f = R_\theta \circ \rho$. Here $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection across the horizontal axis, i.e., $\rho(x, y) := (x, -y)$.

By part (d) we know f is linear, hence can be written as a matrix. The first column of this matrix is $f(1, 0)$ and the second is $f(0, 1)$, so it suffices to figure out what these can be. By part (b), we know that

$$|f(1, 0)| = |(1, 0)| = 1,$$

which means $f(1, 0)$ lies on the unit circle centered at the origin. Thus there exists some θ such that

$$f(1, 0) = (\cos\theta, \sin\theta).$$

Next, by part (c) we have

$$f(1, 0) \cdot f(0, 1) = (1, 0) \cdot (0, 1) = 0,$$

whence $f(1, 0)$ and $f(0, 1)$ are perpendicular. Since $f(0, 1)$ also lies on the unit circle, we deduce that either

$$f(0, 1) = \Big(\cos\big(\theta + \frac{\pi}{2}\big), \sin\big(\theta + \frac{\pi}{2}\big)\Big) = (-\sin\theta, \cos\theta)$$

or

$$f(0, 1) = \Big(\cos\big(\theta - \frac{\pi}{2}\big), \sin\big(\theta - \frac{\pi}{2}\big)\Big) = (\sin\theta, -\cos\theta).$$

Putting all this together, we conclude that there exists $\theta \in \mathbb{R}$ such that either

$$f = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad f = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

In the former situation, $f = R_\theta$; in the latter, $f = R_\theta \circ \rho$ (see **M1-4(a)** above). $\square$

(f) Prove that any distance-preserving map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ can be written as the composition of a translation, a rotation, and (possibly) a reflection. [A *translation* is a map $T_k : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_k(x) := x + k$. I'm asking you to prove that either $\phi = T_k \circ R_\theta$ or $\phi = T_k \circ R_\theta \circ \rho$.]

Consider the function

$$f := T_{-\phi(\mathbf{0})} \circ \phi.$$

From **M1-5(a)**, we know that any translation is distance-preserving. It follows that for any $x, y \in \mathbb{R}^2$, $|f(x) - f(y)| = \big|\big(T_{-\phi(\mathbf{0})} \circ \phi\big)(x) - \big(T_{-\phi(\mathbf{0})} \circ \phi\big)(y)\big| = |\phi(x) - \phi(y)| = |x - y|$; thus f is distance-preserving. Also, $f(\mathbf{0}) = \big(T_{-\phi(\mathbf{0})} \circ \phi\big)(\mathbf{0}) = \phi(\mathbf{0}) - \phi(\mathbf{0}) = \mathbf{0}$. The map f therefore satisfies the hypotheses of part (e), and thus $\exists\theta$ such that $f = R_\theta$ or $f = R_\theta \circ \rho$. Since $\phi = T_{\phi(\mathbf{0})} \circ f$, we conclude that either $\phi = T_{\phi(\mathbf{0})} \circ R_\theta$ or $\phi = T_{\phi(\mathbf{0})} \circ R_\theta \circ \rho$. $\square$