Instructor: Leo Goldmakher

NAME: ______________________________

SECTION: ____________________________

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Midterm Exam 2 – due Tuesday, April 26th

INSTRUCTIONS:
This is a 24-hour midterm. This means that once you begin looking at the midterm, you must cease all activity on the midterm 24 hours later.
This midterm must be turned in *as a hard copy* to the box outside my office by **9pm** on Tuesday. *This is a sharp deadline.* However, if you finish your 24-hour block before Tuesday, I urge you to submit the exam as soon as you're done taking it.

Please print and attach this page as the first page of your submitted exam.

PROBLEM	**GRADE**
1(a)	
1(b)	
2(a)	
2(b)	
2(c)	
2(d)	
2(e)	
Total	

Please read the following statement and sign below:

I understand that the only aids I may use on this exam are my notes from class, my graded problem sets, the solution sets from the problem sets, and the posted lecture summaries. ***No other aids are permitted.*** *In particular, I may not use the internet or books to assist me in any way on this exam, nor may I consult any person (apart from the instructor). I pledge to abide by the Williams honor code.*

SIGNATURE:____

Midterm 2

M2–1 Consider the sequence $1, 2, 5, 12, 29, \ldots$ where $g_1 := 1$, $g_2 := 2$, and $g_{n+1} := 2g_n + g_{n-1}$ for all $n \geq 2$. The goal of this exercise is to adapt the method we used to find an explicit formula for the Fibonacci numbers to this sequence.

(a) Recall that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ generates the Fibonacci numbers. Find a matrix which generates the sequence g_n. Prove that your matrix does so.

(b) Use the matrix you found in (a) and the method from class to determine an explicit (i.e., non-recursive) formula for g_n. [*If you are unable to solve part (a), this part of the problem will not be possible. In this case, instead determine a formula for the top-left entry of* $\begin{pmatrix} 15 & 4 \\ 4 & 0 \end{pmatrix}^n$]

M2–2 Given a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, say with matrix $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define the function $f^t : \mathbb{R}^2 \to \mathbb{R}^2$ (called the *transpose* of f) to be the linear map corresponding to the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. For example, if $f = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $f^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

(a) Prove that for any two linear maps $f, g : \mathbb{R}^2 \to \mathbb{R}^2$, we have $(f \circ g)^t = g^t \circ f^t$.

(b) Prove that $R_\theta^{-1} = R_\theta^t$ for any θ.

(c) Is it true the $f^t = f^{-1}$ for every linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$? If yes, prove it. If not, find a counterexample.

(d) Suppose the singular value decomposition of f is

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta.$$

What is the singular value decomposition of the function $f \circ f^t$?

(e) What's the relationship between the singular values of f and its eigenvalues? Be as precise as you can.