Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Midterm Exam 2 – KEY

M2–1 Consider the sequence $1, 2, 5, 12, 29, \ldots$ where $g_1 := 1$, $g_2 := 2$, and $g_{n+1} := 2g_n + g_{n-1}$ for all $n \geq 2$. The goal of this exercise is to adapt the method we used to find an explicit formula for the Fibonacci numbers to this sequence.

(a) Recall that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ generates the Fibonacci numbers. Find a matrix which generates the sequence g_n. Prove that your matrix does so.

Claim. $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{pmatrix}$ *for all* $n \geq 2$.

Proof. We proceed by induction. Suppose that the claim holds true for some integer $n \geq 2$. I claim that the claim must continue to hold for $n + 1$. Indeed, we have

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2g_{n+1} + g_n & g_{n+1} \\ 2g_n + g_{n-1} & g_n \end{pmatrix}$$

Using the recursive definition of g_n, we deduce that

$$(*) \qquad \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{pmatrix} \implies \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} g_{n+2} & g_{n+1} \\ g_{n+1} & g_n \end{pmatrix}$$

A quick calculation shows that the claim holds for $n = 2$. The relation (*) then implies that the claim also holds for $n = 3$; applying (*) again implies the claim for $n = 4$; etc. Thus, the claim must hold for all $n \geq 2$. $\square$

(b) Use the matrix you found in (a) and the method from class to determine an explicit (i.e., non-recursive) formula for g_n. [*If you are unable to solve part (a), this part of the problem will not be possible. In this case, instead determine a formula for the top-left entry of* $\begin{pmatrix} 15 & 4 \\ 4 & 0 \end{pmatrix}^n$]

First we find the eigenvalues by solving the equation

$$\det\left(\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0.$$

Simplifying the left hand side yields $\lambda^2 - 2\lambda - 1 = 0$; applying the quadratic equation gives the eigenvalues:

$$\lambda_1 = 1 + \sqrt{2} \qquad \lambda_2 = 1 - \sqrt{2}.$$

Solving the equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$$

gives $x = \lambda_1 y$. Taking $y = 1$, we find an eigenvector corresponding to λ_1:

$$\vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}.$$

Similarly, we find that the eigenvector corresponding to λ_2 is

$$\vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}.$$

Letting P denote the change of basis matrix $\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$, we determine the spectral decomposition of $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}.$$

Some computation shows that

$$P^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

We conclude that

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1};$$

multiplying all three matrices together, we can determine all four entries of the resulting matrix. On the other hand, by (a) we know that the bottom left corner is g_n! Thus,

$$g_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

M2–2 Given a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$, say with matrix $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define the function $f^t : \mathbb{R}^2 \to \mathbb{R}^2$ (called the *transpose* of f) to be the linear map corresponding to the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. For example, if $f = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $f^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

(a) Prove that for any two linear maps $f, g : \mathbb{R}^2 \to \mathbb{R}^2$, we have $(f \circ g)^t = g^t \circ f^t$.

Proof. Since f, g are linear, we can write

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Then

$$(f \circ g)^t = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}^t = \begin{pmatrix} ap + br & cp + dr \\ aq + bs & cq + ds \end{pmatrix}$$
$$= \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = g^t \circ f^t.$$ □

(b) Prove that $R_\theta^{-1} = R_\theta^t$ for any θ.

Proof.

$$R_\theta^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^t = R_\theta^t$$ □

(c) Is it true that $f^t = f^{-1}$ for all linear maps $f : \mathbb{R}^2 \to \mathbb{R}^2$? If yes, prove it. If not, find a counterexample.

No, this is false. For example,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^t$$

(d) Suppose the singular value decomposition of f is

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta.$$

What is the singular value decomposition of the function $f \circ f^t$?

From part (a) of this question (as well as associativity of function composition), we see that

$$\left(R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta \right)^t = R_\beta^t \circ \left(R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} \right)^t = R_\beta^t \circ \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix}^t \circ R_\alpha^t.$$

Applying part (b) to simplify this, we deduce

$$f \circ f^t = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta R_\beta^{-1} \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\alpha^{-1} = R_\alpha \begin{pmatrix} k^2 & 0 \\ 0 & \ell^2 \end{pmatrix} R_{-\alpha}$$

(e) What's the relationship between the singular values of f and its eigenvalues? Be as precise as you can.

This is an open-ended question, and there were a number of nice observations people made. Here are the three most common ones.

Proposition 1. *The eigenvalues of $f \circ f^t$ are the squares of the singular values of f.*

Proof. In part (d) of this problem we proved that if the SVD of f is $f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta$, then

$$f \circ f^t = R_\alpha \begin{pmatrix} k^2 & 0 \\ 0 & \ell^2 \end{pmatrix} R_\alpha^{-1}.$$

Problem **6.3(a)** implies that k^2 and ℓ^2 are eigenvalues of $f \circ f^t$. $\square$

Note that this result is useful for determining the singular values of f: instead of finding them directly, we can first find the eigenvalues of $f \circ f^t$ and then use this proposition.

A different observation was about the magnitudes of the eigenvalues and singular values:

Proposition 2. *Suppose k and ℓ are the singular values of f, say with $k \geq \ell \geq 0$. Then $\ell \leq |\lambda| \leq k$ for any eigenvalue λ of f.*

Proof. If λ is an eigenvalue of f, then by definition there exists some eigenvector $\vec{u}_0 \neq \mathbf{0}$ such that

$$f(\vec{u}_0) = \lambda \vec{u}_0$$

Since any rescaling of an eigenvector is still an eigenvector, we may assume that $\vec{u}_0$ is a unit vector. In particular, $|f(\vec{u}_0)| = |\lambda|$. Since k is the length of the major axis and ℓ the length of the minor radius of the ellipse $f(U)$, we see that

$$k := \max_{|\vec{u}|=1} |f(\vec{u})| \qquad \text{and} \qquad \ell := \min_{|\vec{u}|=1} |f(\vec{u})|$$

This immediately implies the claim. $\square$

It turns out that the products of the eigenvalues and the singular values are related.

Proposition 3. *Given a diagonalizable f, let k, ℓ be its singular values and λ_1, λ_2 be its eigenvalues. (If f only has a single eigenvalue, count it twice by setting $\lambda_1 = \lambda_2$.) Then $k\ell = \lambda_1\lambda_2$.*

Proof. Since f is diagonalizable, we can write

$$f = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$$

for some invertible linear map P. On the other hand, the SVD of f is

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta.$$

It follows that

$$\lambda_1\lambda_2 = \det f = k\ell$$

since $\det P^{-1} = \frac{1}{\det P}$ and $\det R_\theta = 1$. $\square$