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Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Problem Set 1 – due Thursday, February 18th

INSTRUCTIONS:

This assignment must be turned in *as a hard copy*, to the box to the left of my office door (215B), by **9pm** sharp. Assignments turned in later than this, but before 5pm on Friday, will also be graded, but the grade will be reduced by one mark. *Assignments submitted later than Friday at 5pm will be returned without being marked.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
1.1	
1.2	
1.3	
1.4	
1.5	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to search for problems or solutions. I also understand that I must write down the final version of my assignment without reference to notes copied from anyone else's speech or written text. I pledge to abide by the Williams honor code.

SIGNATURE: _____

Problem Set 1

1.1 The goal of this problem is to give an alternative proof that $\sqrt{2}$ is irrational. Let

$$\mathcal{A} := \{n \in \mathbb{Z} : n > 0, n\sqrt{2} \in \mathbb{Z}\}.$$

- (a) Prove that if $a \in \mathcal{A}$, then $a(\sqrt{2} - 1) \in \mathcal{A}$.
- (b) Use part (a) to show that the set $\mathcal{A}$ must be empty.
- (c) Use part (b) to explain why $\sqrt{2} \notin \mathbb{Q}$.

1.2 The goal of this exercise is to explore how robust our proofs of irrationality are.

- (a) Adapt the proof from class that $\sqrt{2} \notin \mathbb{Q}$ to prove that $\sqrt{7} \notin \mathbb{Q}$.
- (b) Adapt the approach from problem **1.1** to give a different proof that $\sqrt{7} \notin \mathbb{Q}$.

1.3 Given $\alpha, \beta \notin \mathbb{Q}$ and $q \in \mathbb{Q}$.

- (a) Must it be true that $q + \alpha \notin \mathbb{Q}$? If so, prove it. If not, give a counterexample (i.e. give explicit choices of q and α such that $q + \alpha \in \mathbb{Q}$).
- (b) Must it be true that $q\alpha \notin \mathbb{Q}$? If so, prove it. If not, give a counterexample.
- (c) Must it be true that $\alpha + \beta \notin \mathbb{Q}$? Justify your response with a proof or counterexamples.
- (d) Must it be true that $\alpha\beta \notin \mathbb{Q}$? Justify your response with a proof or counterexamples.

1.4 Let $[x]$ denote the largest integer smaller than x ; this is called the *floor of x* . For example, $[\pi] = 3$, $[7] = 7$, and $[-\pi] = -4$. The goal of this problem is to explore the distribution of rationals and irrationals within $\mathbb{R}$. Note that the word *between* is used in a strict sense: x is between a and b means $a < x < b$.

- (a) Given $b > a + 1$, prove that there exists an integer between a and b .
- (b) Given $x < y$, prove that there exists a rational number between x and y . [*Hint: use part (a).*]
- (c) Prove that there exists an irrational number between 0 and 1. [*Hint: don't think too hard.*]
- (d) Prove that between any two rational numbers there exists an irrational number. [*Hint: use part (c).*]
- (e) Prove that between any two real numbers there is an irrational. [*Hint: use parts (b) and (d).*]

1.5 Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following three properties:

- (i) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
 - (ii) $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.
 - (iii) $f(1) \neq 0$.
- (a) Prove that $f(1) = 1$.
 - (b) Prove that $f(x) > 0$ whenever $x > 0$. [*Hint: start by proving that $f(x) \neq 0$ whenever $x \neq 0$.*]
 - (c) Prove that $f(x) > f(y)$ whenever $x > y$.
 - (d) Prove that $f(x) = x$ for all $x \in \mathbb{R}$. [*Hint: Use 1.4(b)*]