Instructor: Leo Goldmakher

NAME: ______________________________

SECTION: ____________________________

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Problem Set 6 – due Thursday, April 21st

INSTRUCTIONS:

This assignment must be turned in *as a hard copy* to the mailbox of your TA (on your left as soon as you enter Bronfman from Science Quad, labelled by last name), by **9pm** sharp. Assignments turned in later than this, but before 5pm on Friday, will also be graded, but the grade will be reduced by one mark. *Assignments submitted later than Friday at 5pm will be returned without being marked.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
6.1	
6.2	
6.3	
6.4	
6.5	
6.6	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to search for problems or solutions. I also understand that I must write down the final version of my assignment without reference to notes copied from anyone else's speech or written text. I pledge to abide by the Williams honor code.

SIGNATURE:____

Problem Set 6

6.1 Recall (from Lecture 21) the notion of an *equivalence relation.* Decide whether each of the following binary relations is an equivalence relation. If it is, prove it. If not, give an example of how it fails.

(a) $\sim$ (matrix similarity)

(b) $\leq$ (less than or equal to)

(c) $\approx$ (given two sets $A, B \subseteq \mathbb{Z}$ we write $A \approx B$ if and only if A and B differ by finitely many elements. For example, $\{0, 1, 2, 3, \ldots\} \approx \{1, 2, 3, \ldots\}$ since they differ by one element, while $\{1, 2, 3, 4, \ldots\} \not\approx \{2, 4, 6, \ldots\}$ since they differ by infinitely many elements.)

6.2 Suppose f and g are nonsingular linear maps from $\mathbb{R}^2 \to \mathbb{R}^2$.

(a) Show by example that fg might not equal gf.

(b) Prove that $fg \sim gf$ (matrix similarity).

6.3 Suppose P is a nonsingular linear map, and that $f = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$.

(a) Prove that λ_1 and λ_2 are eigenvalues of f.

(b) Find (with proof) an eigenvector corresponding to λ_1?

6.4 For each of the following linear functions, (i) determine all eigenvalues, (ii) for each eigenvalue, find a corresponding eigenvector of unit length, and (iii) if possible, write down a spectral decomposition of f.

(a) $f = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$

(b) $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $h = \begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$

(d) $k = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

(e) f^2, where f is the function from part (a) of this question.

6.5 Let f_n denote the n^{th} Fibonacci number (with $f_1 = f_2 = 1$).

(a) Determine $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ [*Hint: How big is* $\frac{1-\sqrt{5}}{2}$*?*]

(b) Evaluate $f_n^2 + f_{n+1}^2$ for $n = 1, 2, 3, 4$. Conjecture a formula.

(c) Prove your conjectured formula. [*Hint: consider* $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{2n}$]

6.6 (**Bonus**) Prove that any positive integer can be written as the sum of distinct Fibonacci numbers, no two of which are consecutive. For example, $16 = f_4 + f_7$. (In fact, every positive integer has a unique representation in this form!)