Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Problem Set 1 – KEY

1.1 The goal of this problem is to give an alternative proof that $\sqrt{2}$ is irrational. Let

$$\mathcal{A} := \{n \in \mathbb{Z} : n > 0,\ n\sqrt{2} \in \mathbb{Z}\}.$$

(a) Prove that if $a \in \mathcal{A}$, then $a(\sqrt{2}-1) \in \mathcal{A}$.

First, since $1 < 2 < 4$, we deduce that $1 < \sqrt{2} < 2$. In particular,

$$0 < \sqrt{2} - 1 < 1. \qquad (*)$$

Now suppose $a \in \mathcal{A}$. By definition of $\mathcal{A}$, this means

- $a > 0$;
- $a \in \mathbb{Z}$; and
- $a\sqrt{2} \in \mathbb{Z}$.

We now verify that the same hold true of $a(\sqrt{2}-1)$:

- From above, we know $a > 0$ and $\sqrt{2} - 1 > 0$, so $a(\sqrt{2}-1) > 0$.
- $a(\sqrt{2}-1) = a\sqrt{2} - a \in \mathbb{Z}$, since both a and $a\sqrt{2}$ are integers.
- $a(\sqrt{2}-1)\sqrt{2} = 2a - a\sqrt{2} \in \mathbb{Z}$ as above.

Thus $a(\sqrt{2}-1) \in \mathcal{A}$, since it satisfies all the membership requirements. QED

(b) Use part (a) to show that the set $\mathcal{A}$ must be empty. [*Hint: Start by using* (a) *to prove that* $1 \notin \mathcal{A}$.]

Suppose $\mathcal{A}$ weren't empty, and let ℓ denote the smallest element of $\mathcal{A}$. By part (a), we would have $\ell(\sqrt{2}-1) \in \mathcal{A}$ as well. But by the inequality (*) above, $\ell(\sqrt{2}-1)$ is smaller than ℓ. Contradiction! QED

(c) Use part (b) to explain why $\sqrt{2} \notin \mathbb{Q}$.

Suppose $\sqrt{2} \in \mathbb{Q}$. Then we could write $\sqrt{2} = \frac{a}{n}$ for some positive integers a and n. But then this would imply that $n \in \mathcal{A}$, contradicting part (b). Thus $\sqrt{2} \notin \mathbb{Q}$. QED

1.2 The goal of this exercise is to explore how robust our proofs of irrationality are.

(a) Adapt the proof from class that $\sqrt{2} \notin \mathbb{Q}$ to prove that $\sqrt{7} \notin \mathbb{Q}$.

Suppose $\sqrt{7} \in \mathbb{Q}$. We will show that this leads to a contradiction.
Write $\sqrt{7} = \frac{a}{b}$, where we may safely assume that $\frac{a}{b}$ is a reduced fraction (i.e., that a and b are both positive integers with no common factor greater than 1; if not, reduce the fraction!). It follows that

$$a^2 = 7b^2, \tag{$\dagger$}$$

which implies that a^2 is a multiple of 7. I now claim:

Lemma 1. *Given $n \in \mathbb{Z}$. If n^2 is a multiple of 7, then so is n.*

I'll prove this below, but first let's use it to complete the current proof. Since a^2 is a multiple of 7, the lemma implies that a must be a multiple of 7. In other words, there exists some $k \in \mathbb{Z}$ such that $a = 7k$. Plugging this into ($\dagger$) yields $b^2 = 7k^2$. Thus, b^2 is a multiple of 7; the lemma implies that b must also be a multiple of 7. Thus both a and b are multiples of 7, contradicting that the fraction $\frac{a}{b}$ was reduced.
QED

Proof of Lemma. Given $n \in \mathbb{Z}$ such that n^2 is a multiple of 7. Let $q := \left[\frac{n}{7}\right]$ (the floor function; see problem **1.4** below). By definition of the floor function, $q \in \mathbb{Z}$ and

$$q \leq \frac{n}{7} < q + 1.$$

It follows that $7q \leq n < 7q + 7$; since both n and q are integers, we deduce that

$$n = 7q + r$$

for some $r = 0, 1, 2, 3, 4, 5$, or 6. Thus,

$$r^2 = n^2 - 7(7q^2 + 2qr)$$

which is a multiple of 7. It is easy to verify by trial and error that the only possible value of r such that r^2 is a multiple of 7 is $r = 0$. But this means that $n = 7q$, i.e., that n is a multiple of 7. □

(b) Adapt the approach from problem **1.1** to give a different proof that $\sqrt{7} \notin \mathbb{Q}$.

Let $\mathcal{B} := \{n > 0 : n \in \mathbb{Z}, n\sqrt{7} \in \mathbb{Z}\}$, and let b denote the smallest element of $\mathcal{B}$. But then $b(\sqrt{7} - 2)$ is a smaller element of $\mathcal{B}$, contradicting the minimality of b. This shows that $\mathcal{B}$ must be empty, whence $\sqrt{7} \notin \mathbb{Q}$ by the same argument as in **1.1(c)**.
QED

1.3 Given $\alpha, \beta \notin \mathbb{Q}$ and $q \in \mathbb{Q}$.

(a) Must it be true that $q + \alpha \notin \mathbb{Q}$? If so, prove it. If not, give a counterexample (i.e. give explicit choices of q and α such that $q + \alpha \in \mathbb{Q}$).

We first prove a useful

Lemma 2. *For any $a, b \in \mathbb{Q}$ we have $a \pm b, ab \in \mathbb{Q}$. If $b \neq 0$ we also have $a/b \in \mathbb{Q}$.*

Proof. Write $a = \frac{a_1}{a_2}$ and $b = \frac{b_1}{b_2}$ with all the $a_i, b_i \in \mathbb{Z}$. The assertions are now straightforward to verify. □

Armed with this, it's easy to prove that $q + \alpha \notin \mathbb{Q}$. Indeed, if $q + \alpha = r \in \mathbb{Q}$, then $\alpha = r - q \in \mathbb{Q}$ by the lemma.

(b) Must it be true that $q\alpha \notin \mathbb{Q}$? If so, prove it. If not, give a counterexample.

Usually, but not always. If $q = 0$ (which is rational: $0 = \frac{0}{1}$) then $q\alpha \in \mathbb{Q}$. On the other hand:

Proposition 3. *If $q \in \mathbb{Q}$ and $q \neq 0$, and $\alpha \notin \mathbb{Q}$, then $q\alpha \notin \mathbb{Q}$.*

Proof. If $q\alpha = r \in \mathbb{Q}$, then (since $q \neq 0$) we would have $\alpha = r/q \in \mathbb{Q}$, contradicting the irrationality of α. □

(c) Must it be true that $\alpha + \beta \notin \mathbb{Q}$? Justify your response with a proof or counterexamples.

Not necessarily. For example, $-\sqrt{2} + \sqrt{2} \in \mathbb{Q}$, even though $\pm\sqrt{2} \notin \mathbb{Q}$. On the other hand, often the sum of two irrationals *is* irrational; it's a fun exercise to prove that $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

(d) Must it be true that $\alpha\beta \notin \mathbb{Q}$? Justify your response with a proof or counterexamples.

Not necessarily: $\sqrt{2} \cdot \sqrt{2} \in \mathbb{Q}$. On the other hand, one can show (using either of the two methods we've seen) that $\sqrt{2} \cdot \sqrt{3} \notin \mathbb{Q}$.

1.4 Let $[x]$ denote the largest integer smaller than x; this is called the *floor of* x. For example, $[\pi] = 3$, $[7] = 7$, and $[-\pi] = -4$. The goal of this problem is to explore the distribution of rationals and irrationals within $\mathbb{R}$. Note that the word *between* is used in a strict sense: x is between a and b means $a < x < b$.

(a) Given $b > a + 1$, prove that there exists an integer between a and b. [*Hint: write down an explicit formula for such an integer.*]

I claim that $a < [a] + 1 < b$. Indeed, from the definition of the floor function we deduce

$$[a] \leq a < [a] + 1.$$

It follows that $a < [a] + 1 \leq a + 1 < b$, as claimed. QED

(b) Given $x < y$, prove that there exists a rational number between x and y. [*Hint: use part* (a).]

Pick any integer $b > \frac{1}{y-x}$. (In particular, $b > 0$.) Then $by > bx + 1$, so by part (a) there must exist some integer a such that $bx < a < by$. But this implies that

$$x < \frac{a}{b} < y.$$

QED

(c) Prove that there exists an irrational number between 0 and 1. [*Hint: it suffices to give a specific example. Of course, you must still prove that your example is irrational.*]

Consider $\sqrt{2}/2$. By the Lemma from **1.3(a)**, this must be irrational. From **1.1(a)** we know

$$\frac{1}{2} < \frac{\sqrt{2}}{2} < 1.$$

(d) Prove that between any two rational numbers there exists an irrational number. [*Hint: use part* (c).]

Suppose $A < B$ are two rationals. We know by (c) the existence of an irrational number α such that $0 < \alpha < 1$. Note that $(B - A)\alpha + A$ is irrational by **1.3(a)** and **(b)**. Now $0 < (B - A)\alpha < B - A$, whence $A < (B - A)\alpha + A < B$; we've found an irrational between A and B. QED

(e) Prove that between any two real numbers there is an irrational. [*Hint: use parts* (b) *and* (d).]

Given two reals x and y. By (b), there exists a rational number A with $x < A < y$. Again by (b), there exists a rational number B with $A < B < y$. By (d), there is an irrational α between A and B. It follows that $x < \alpha < y$. QED

1.5 Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies the following three properties:

(i) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(ii) $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

(iii) $f(1) \neq 0$.

(a) Prove that $f(1) = 1$.

By (ii),

$$f(1)^2 = f(1),$$

whence $f(1) = 0$ or 1. By (iii), $f(1) = 1$. QED

(b) Prove that $f(x) > 0$ whenever $x > 0$. [*Hint: As a warm-up, prove that* $f(\sqrt{2}) = \pm\sqrt{2}$.]

Given $x > 0$. Then, by (ii), we have

$$f(x) = f(\sqrt{x})^2 \geq 0.$$

It therefore suffices to prove that $f(x) \neq 0$. By (ii) and (a),

$$f(x)f(1/x) = f(1) = 1,$$

whence $f(x) \neq 0$. QED

(c) Prove that $f(x) > f(y)$ whenever $x > y$. [*Hint: use part* (b).]

Given $x > y$. By (b) we know that

$$f(x - y) > 0.$$

Additivity of f implies that $f(x - y) = f(x) - f(y)$, whence

$$f(x) > f(y).$$

QED

(d) Prove that $f(x) = x$ for all $x \in \mathbb{R}$. [*Hint: What happens if this isn't true? Use* **1.4(b)**]

Suppose the statement is false. Then there exists some x such that $f(x) \neq x$. It follows that one of x or $f(x)$ must be larger than the other. Say $x < f(x)$ (the proof of the other case is exactly the same). By **1.4(b)**, there is a rational number q such that

$$x < q < f(x). \tag{‡}$$

From class and part (a), we know that $f(q) = qf(1) = q$. Thus (‡) becomes

$$x < q = f(q) < f(x). \tag{**}$$

Since $x < q$, part (c) implies $f(x) < f(q)$. But this contradicts (**)! Thus, no such number x can exist, and the proof is done. QED