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MATH 250 : LINEAR ALGEBRA

Problem Set 3 – KEY

3.1 Compute each of the following.

(a) $\begin{pmatrix} 2 & 3 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 22 \\ 19 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 8 \\ 10 \end{pmatrix}$

(c) $R_{3\pi/4}(2,1)$

From our formula for R_θ, we deduce $R_{3\pi/4} = \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$. Thus

$$R_{3\pi/4}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}.$$

For consistency of notation, we can write this in the form

$$\left(-\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

(d) $\rho\big(R_{\pi/3}(3,4)\big)$, where $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection across the horizontal axis.

First, we figure out where the rotation takes $(3,4)$:

$$\begin{aligned} R_{\pi/3}\begin{pmatrix} 3 \\ 4 \end{pmatrix} &= \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} - 2\sqrt{3} \\ 2 + \frac{3}{2}\sqrt{3} \end{pmatrix}. \end{aligned}$$

Reflecting this across the horizontal axis yields the point

$$\left(\frac{3}{2} - 2\sqrt{3},\ -2 - \frac{3}{2}\sqrt{3}\right)$$

3.2 Below are matrices corresponding to functions mapping $\mathbb{R}^2$ to $\mathbb{R}^2$. Describe each function geometrically. (For example, a geometric description of R_θ might be: it rotates the plane counterclockwise around the origin by angle θ.)

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

This function reflects $\begin{pmatrix} x \\ y \end{pmatrix}$ over the line $y = x$.

(b) $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 5y \end{pmatrix}$$

This function stretches by a factor of 2 horizontally and by a factor of 5 vertically.

(c) $\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$

$$\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x \\ 5y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2x \\ 5y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The above calculation shows that this function performs the same operation as the one in part (b), and then reflects the result over the vertical axis.

(d) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix} &= \sqrt{2}\begin{pmatrix} \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{pmatrix} \\
&= \sqrt{2}\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \sqrt{2}R_{\pi/4}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= R_{\pi/4}\begin{pmatrix} x\sqrt{2} \\ y\sqrt{2} \end{pmatrix} \\
&= R_{\pi/4} \circ \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

Thus, this map stretches in all directions by a factor of $\sqrt{2}$, and then rotates counterclockwise about the origin by $\pi/4$.

3.3 Determine the matrix of $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) := (2x - 3y, x + y)$.

I claim $F = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$. This is easily verified:

$$\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 3y \\ x + y \end{pmatrix} = F \begin{pmatrix} x \\ y \end{pmatrix}$$

3.4 Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear function with matrix $\begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}$, and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear function with matrix $\begin{pmatrix} 6 & -1 \\ -8 & 7 \end{pmatrix}$. Consider the functions $h : \mathbb{R}^2 \to \mathbb{R}^2$ and $k : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h(x, y) := f(x, y) + g(x, y) \qquad \text{and} \qquad k(x, y) := f\big(g(x, y)\big).$$

(a) Determine $h(1, 0)$, $h(0, 1)$, $k(1, 0)$, and $k(0, 1)$.

From class we know that the first column of the matrix is where it maps $(1, 0)$ and the second column is where it maps $(0, 1)$. We therefore see that $f(1, 0) = (3, 2)$, $f(0, 1) = (-5, 4)$, $g(1, 0) = (6, -8)$, and $g(0, 1) = (-1, 7)$. From this we immediately deduce

$$h(1, 0) = (9, -6) \qquad \text{and} \qquad h(0, 1) = (-6, 11)$$

Next we find the outputs of k:

$$k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 58 \\ -20 \end{pmatrix}$$

$$k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 7 \end{pmatrix} = \begin{pmatrix} -38 \\ 26 \end{pmatrix}$$

(b) Prove that h is linear, and determine the matrix of h.

To verify that h is linear we need to show that it's additive and scales.

Additivity. Given any $x, y \in \mathbb{R}^2$, we have

$$h(x + y) = f(x + y) + g(x + y) = f(x) + f(y) + g(x) + g(y) = h(x) + h(y).$$

Scaling. Given any $x \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we have

$$h(\alpha\, x) = f(\alpha\, x) + g(\alpha\, x) = \alpha f(x) + \alpha g(x) = \alpha h(x).$$

Thus, h is linear. We can therefore use part (a) to tell us the columns of the matrix of h:

$$h = \begin{pmatrix} 9 & -6 \\ -6 & 11 \end{pmatrix}$$

(c) Prove that k is linear, and determine the matrix of k.

First we verify that k is linear:

Additivity. Given $x, y \in \mathbb{R}^2$, we have

$$k(x+y) = f\big(g(x+y)\big) = f\big(g(x)+g(y)\big) = f\big(g(x)\big) + f\big(g(y)\big) = k(x) + k(y).$$

Scaling. Given $x \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we have

$$k(\alpha\, x) = f\big(g(\alpha\, x)\big) = f\big(\alpha\, g(x)\big) = \alpha f\big(g(x)\big) = \alpha k(x).$$

Thus k is linear. Using what we found in part (a) yields

$$k = \begin{pmatrix} 58 & -38 \\ -20 & 26 \end{pmatrix}$$

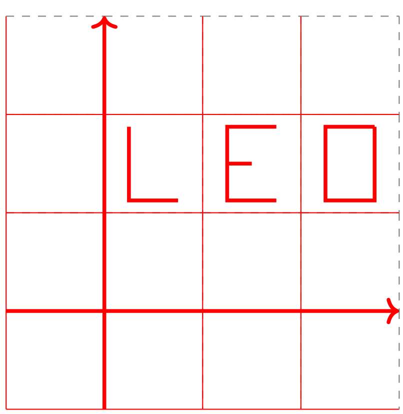
3.5 Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear such that $T(0,1) = (0,1)$ and $T(1,0) = \big(\frac{1}{4}, \frac{\sqrt{3}}{4}\big)$.

(a) Determine the matrix of T.

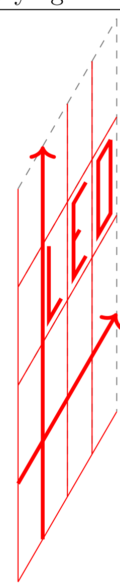
We know from class that the first column of the matrix of T is $T(1,0)$ and the second column of the matrix of T is $T(0,1)$. In other words,

$$T = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{\sqrt{3}}{4} & 1 \end{pmatrix}$$

(b) Let $S_1 := \{(x,y) : 0 \le x \le 1, 1 \le y \le 2\}$, $S_2 := \{(x,y) : 1 \le x \le 2, 1 \le y \le 2\}$, and $S_3 := \{(x,y) : 2 \le x \le 3, 1 \le y \le 2\}$. Carefully write the first letter of your first name in S_1, the second letter of your first name in S_2, and the third letter of your first name in S_3. (See below for illustration.) What would these three letters look like after applying T to them? Draw a clear picture.



Leo's name before applying T


Leo's name after applying T

3.6 Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Show that if $ad - bc = 0$, then there exists a line $\mathcal{L}$ passing through the origin such that all outputs of f lie on $\mathcal{L}$.

First observe that if $b = d = 0$, then for any x, y we have $f(x, y) = (ax, cx)$. All these points lie on a single line passing through the origin: if $a = 0$ they're all on the vertical line through the origin, and if $a \neq 0$ they're all on the line of slope $\frac{c}{a}$ through the origin. Thus if $b = d = 0$, the proof is finished.

Thus it suffices to prove the result when one of b or d is nonzero. Note that for any x, y, the coordinates of the point $f(x, y)$ are related to one another:

$$d(ax + by) = adx + bdy = bcx + bdy = b(cx + dy)$$

(we've used the hypothesis that $ad - bc = 0$). Once again we see that all the points $f(x, y)$ lie on a single line: if $b = 0$ all the points $(ax + by, cx + dy)$ lie on the vertical line through the origin, while if $b \neq 0$ they all lie on the line of slope $\frac{d}{b}$ through the origin. QED

(b) Conversely, show that if there exists a line $\mathcal{L}$ such that $f(x, y)$ is on $\mathcal{L}$ for every (x, y), then $ad - bc = 0$.

Suppose all of the points $f(x, y) = (ax + by, cx + dy)$ lie on a single line $\mathcal{L}$ through the origin. If $\mathcal{L}$ is the vertical line, this implies that

$$ax + by = 0 \qquad \text{for all real numbers } x, y.$$

In particular, taking $x = 1$ and $y = 0$ implies that $a = 0$; if we instead take $x = 0$ and $y = 1$ we deduce $b = 0$. It immediately follows that $ad - bc = 0$.

Now suppose instead that $\mathcal{L}$ is not vertical. In this case $\mathcal{L}$ has a slope, say, m, and we see that

$$cx + dy = m(ax + by) \qquad \text{for all real numbers } x, y.$$

Taking $x = 1$ and $y = 0$ yields $c = am$; taking $x = 0$ and $y = 1$ gives $d = bm$. It follows that $ad - bc = a(bm) - b(am) = 0$ as claimed. QED