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MATH 250 : LINEAR ALGEBRA

Problem Set 4 – KEY

4.1 Suppose $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ are linear maps. *Without using matrices*, prove that $f \circ (g + h) = f \circ g + f \circ h$.

Pick an $x \in \mathbb{R}^2$. Then

$$\begin{aligned}\big(f \circ (g+h)\big)(x) &= f\big(g(x) + h(x)\big) \\ &= f\big(g(x)\big) + f\big(h(x)\big) \qquad \text{(by additivity)} \\ &= (f \circ g)(x) + (f \circ h)(x) \\ &= (f \circ g + f \circ h)(x).\end{aligned}$$

Thus the two functions $f \circ (g + h)$ and $f \circ g + f \circ h$ agree on every input. It follows that they are the same function. QED

Note: Simply saying f is additive does not suffice, since additivity applies to the sum of points as inputs, *not* to the sum of functions.

4.2 Prove that a singular linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is not invertible.

Given a singular linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Since it is linear, we can write f as a matrix, say,

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

since f is singular, $ad - bc = 0$. I claim that the preimage of $\mathbf{0}$ has at least two elements; this immediately implies that f is not invertible.

First, if $a = b = c = d = 0$, then both $\mathbf{0}$ and $(1, 1)$ live in $f^{-1}(\mathbf{0})$, hence f is noninvertible. Thus, we may assume that at least one of a, b, c, d are nonzero.

Next, observe that

$$f(b, -a) = \mathbf{0} = f(d, -c).$$

One of $(b, -a)$ or $(d, -c)$ must be different from $\mathbf{0}$. It follows that $\#f^{-1}(\mathbf{0}) \geq 2$, and we conclude. QED

4.3 The *zero function* is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, i.e., the function mapping $\mathbb{R}^2 \to \mathbb{R}^2$ which outputs $\mathbf{0}$ for all inputs. Now suppose $f \circ g$ is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$, where neither of $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are the zero function. Must f be linear? If so, prove it. If not, produce a counterexample.

No, f is not necessarily linear. For example, consider

$$f(x, y) := \begin{cases} (\frac{1}{x}, 0) & \text{if } x \neq 0 \\ (0, 0) & \text{otherwise.} \end{cases}$$

f is not additive – for example, $f(1, 0) + f(1, 0) = (2, 0) \neq f(2, 0)$ – so it is also not linear. However, $f \circ f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is linear.

4.4 Given $f : A \to B$ a function and $a \in A$. (Note: f is not necessarily linear!) What can you say about $f\big(f^{-1}(f(a))\big)$? Be as specific as you can, and justify your answer.

I claim that

$$f\big(f^{-1}(f(a))\big) = f(a).$$

By definition,

$$f^{-1}(f(a)) = \{x \in A : f(x) = f(a)\}.$$

In particular, $a \in f^{-1}(f(a))$, so $f^{-1}(f(a)) \neq \emptyset$. Thus we have

$$\begin{aligned} f\big(f^{-1}(f(a))\big) &= f\big(\{x \in A : f(x) = f(a)\}\big) \\ &= \{f(x) : x \in A, f(x) = f(a)\} \\ &= \{f(a)\} \\ &= f(a). \end{aligned}$$

QED

4.5 Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a nonsingular linear map with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) *Without using matrices*, prove that $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map.

Given $x, y \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$.

Additivity. Let $p := f^{-1}(x)$ and $q := f^{-1}(y)$. (p and q exist and are well-defined because f is invertible.) By definition, $f(p) = x$ and $f(q) = y$, whence

$$\begin{aligned} f^{-1}(x + y) &= f^{-1}\big(f(p) + f(q)\big) \\ &= f^{-1}\big(f(p + q)\big) \\ &= p + q \qquad \text{(since } f \text{ is invertible)} \\ &= f^{-1}(x) + f^{-1}(y). \end{aligned}$$

Scaling. Let $p := f^{-1}(x)$. Then $f(p) = x$, whence $\alpha x = \alpha f(p) = f(\alpha p)$. Thus,

$$f^{-1}(\alpha x) = f^{-1}\big(f(\alpha p)\big) = \alpha p = \alpha f^{-1}(x).$$

It follows that f^{-1} is linear. QED

(b) What is the matrix of f^{-1}? Show your work.

Note that

$$f(-b, a) = (0, ad - bc)$$
$$f(d, -c) = (ad - bc, 0)$$

Thus, since f is invertible, we have

$$f^{-1}(ad - bc, 0) = (d, -c)$$
$$f^{-1}(0, ad - bc) = (-b, a)$$

Since f^{-1} is linear and nonsingular, we deduce

$$f^{-1}(1, 0) = \frac{1}{ad - bc}(d, -c)$$
$$f^{-1}(0, 1) = \frac{1}{ad - bc}(-b, a)$$

Finally, since f^{-1} is linear, we know it can be written as a matrix. Moreover, the first column of this matrix is given by $f^{-1}(1, 0)$, and the second column by $f^{-1}(0, 1)$. Hence

$$\boxed{f^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

4.6 DO NOT use a computer or calculator for this exercise!

In each of the following examples, determine (i) the matrix of $(f \circ g)$, (ii) the matrix of $(g \circ f)$, (iii) the matrix of f^{-1}. If the matrix of f^{-1} does not exist, carefully explain (with suitable examples) why not.

(a) $f = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$(f \circ g) = \begin{pmatrix} 3 & 1 \\ 7 & 3 \end{pmatrix}$$
$$(g \circ f) = \begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix}$$
$$f^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

(b) $f = \begin{pmatrix} 2 & -6 \\ -3 & 9 \end{pmatrix}$, $g = \begin{pmatrix} 2 & 5 \\ 4 & -1 \end{pmatrix}$

$$(f \circ g) = \begin{pmatrix} -20 & 16 \\ 30 & -24 \end{pmatrix}$$
$$(g \circ f) = \begin{pmatrix} -11 & 33 \\ 11 & -33 \end{pmatrix}$$

f is not invertible. Indeed,

$$f(3, 1) = \mathbf{0} = f(\mathbf{0}),$$

whence $\# f^{-1}(\mathbf{0}) \geq 2$.

(c) $f = \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix}$, $g = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$

$$(f \circ g) = \begin{pmatrix} 3 & 3 \\ 0 & -8 \end{pmatrix}$$
$$(g \circ f) = \begin{pmatrix} 3 & 10 \\ 0 & -8 \end{pmatrix}$$
$$f^{-1} = \begin{pmatrix} 1 & 1/2 \\ 0 & -1/4 \end{pmatrix}$$

(d) $f = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $g = \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix}$, $ad \neq 0$

$$(f \circ g) = \begin{pmatrix} ak & 0 \\ 0 & d\ell \end{pmatrix}$$
$$(g \circ f) = \begin{pmatrix} ak & 0 \\ 0 & d\ell \end{pmatrix}$$
$$f^{-1} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/d \end{pmatrix}$$

4.7 This exercise explores what linear maps do to triangles. Throughout, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map.

(a) Given $A, B \in \mathbb{R}^2$, the notation $\overline{AB}$ denotes the line segment whose endpoints are A and B. Prove that $f(\overline{AB}) = \overline{f(A)f(B)}$. In particular, a linear map sends line segments to line segments.

Let $A = (x_1, x_2)$ and $B = (y_1, y_2)$. Then

$$\overline{AB} = \big\{\big((1-\alpha)x_1 + \alpha y_1, (1-\alpha)x_2 + \alpha y_2\big) : 0 \leq \alpha \leq 1\big\}.$$

By linearity we therefore have

$$\begin{aligned} f\big(\overline{AB}\big) &= \big\{f\big((1-\alpha)x_1 + \alpha y_1, (1-\alpha)x_2 + \alpha y_2\big) : 0 \leq \alpha \leq 1\big\} \\ &= \big\{f\big((1-\alpha)x_1, (1-\alpha)x_2\big) + f(\alpha y_1, \alpha y_2) : 0 \leq \alpha \leq 1\big\} \\ &= \big\{(1-\alpha)f(x_1, x_2) + \alpha f(y_1, y_2) : 0 \leq \alpha \leq 1\big\} \\ &= \big\{(1-\alpha)f(A) + \alpha f(B) : 0 \leq \alpha \leq 1\big\} \\ &= \overline{f(A)f(B)}. \end{aligned}$$

QED

(b) Consider a triangle $\triangle ABC$ in the plane. What can you say about the shape of the image of $\triangle ABC$ under f? [*Hint: f might be singular or nonsingular.*]

There are three possibilities:

- $f(\triangle ABC)$ is a point. For example, this is the case for $f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $f(\triangle ABC)$ is a line segment. For example, this is the case for $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
- $f(\triangle ABC)$ is a triangle. Indeed, by part (a) we know that

  $$f(\overline{AB}) = \overline{f(A)f(B)} \qquad f(\overline{AC}) = \overline{f(A)f(C)} \qquad f(\overline{BC}) = \overline{f(B)f(C)}.$$

  It follows that

  $$f(\triangle ABC) = \triangle f(A)f(B)f(C).$$

  We must have that $f(A)$, $f(B)$, and $f(C)$ are not collinear, else the output would be a line. Thus, the three points form an honest triangle.

4.8 Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc < 0$, what does this tell you about the geometric effect the matrix has on the plane? Try to describe this as precisely as you can. [*Hint: play around with what the matrix does to a triangle.*]

If $ad - bc < 0$, then the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ changes the chirality of the plane (i.e., it flips the plane over).