Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Problem Set 7 – KEY

7.1 Let V be a vector space.

(a) Given $\vec{v} \in V$, prove that $\vec{v}$ has a unique additive inverse.

Suppose $\vec{w}$ and $\vec{u}$ are both additive inverses of $\vec{v}$. Then by associativity and the definition of additive inverse, we have

$$\vec{w} = \vec{0} + \vec{w} = (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) = \vec{u} + \vec{0} = \vec{u}$$

Thus $\vec{v}$ has a unique additive inverse. □

(b) Prove that $-1 \cdot \vec{v} = -\vec{v}$.

We have

$$\begin{aligned} -1 \cdot \vec{v} &= -1 \cdot \vec{v} + \vec{0} \\ &= -1 \cdot \vec{v} + (\vec{v} - \vec{v}) \\ &= (-1 \cdot \vec{v} + 1 \cdot \vec{v}) - \vec{v} \\ &= (-1 + 1) \cdot \vec{v} - \vec{v} \\ &= 0 \cdot \vec{v} - \vec{v}. \end{aligned}$$

In lecture we proved that $0 \cdot \vec{v} = \vec{0}$, and the claim follows. □

7.2 Problem **2.2** from Chapter 1 of the textbook. (Of course, you must justify your answers with proof or counterexample.)

(a) **True.** Let S be a set of vectors containing $\vec{0}$, and set $S' := S \setminus \{\vec{0}\}$. Then the following nontrivial linear combination gives the zero vector:

$$\vec{0} + \sum_{\vec{v} \in S'} 0 \cdot \vec{v} = \vec{0}$$

This proves that S is linearly dependent.

(b) **False.** A basis is linearly independent by the Fundamental Property of Bases (proved in class), hence cannot contain $\vec{0}$ by part (a). More simply, here's a counterexample to the claim: we proved in class that $\{(1,1),(1,-1)\}$ is a basis of $\mathbb{R}^2$.

(c) **False.** $\{(1,0),(2,0)\}$ is linearly dependent, but $\{(1,0)\}$ is linearly independent.

(d) **True.** Given $\mathcal{L}$ a set of linearly independent vectors and $\mathcal{L}' \subseteq \mathcal{L}$. Suppose $\sum_{\vec{\ell} \in \mathcal{L}'} \alpha_{\vec{\ell}} \vec{\ell} = \vec{0}$. Then $\sum_{\vec{v} \in \mathcal{L} \setminus \mathcal{L}'} 0 \cdot \vec{v} + \sum_{\vec{\ell} \in \mathcal{L}'} \alpha_{\vec{\ell}} \vec{\ell} = \vec{0}$. Since $\mathcal{L}$ is linearly independent, all coefficients appearing above must be 0, whence $\alpha_{\vec{\ell}} = 0$ for every $\vec{\ell} \in \mathcal{L}'$. We've therefore shown that the only linear combination of $\mathcal{L}'$ giving $\vec{0}$ is the trivial linear combination, which implies that $\mathcal{L}'$ is linearly independent.

(e) **False.** In the vector space $\mathbb{R}^2$ we have $2 \cdot (1,0) + (-2,0) = (0,0)$.

7.3 Let $M_{2\times 2}(\mathbb{R})$ denote the space of 2×2 matrices with real entries. What is the dimension of $M_{2\times 2}(\mathbb{R})$? Prove it. [*Hint: First find a basis. Then prove it's a basis. This gives you the dimension.*]

I claim that $\dim M_{2\times 2}(\mathbb{R}) = 4$. This is an immediate consequence of the following

Claim. *Let* $\vec{v}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\vec{v}_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{v}_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, *and* $\vec{v}_4 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Then* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ *is a basis of* $M_{2\times 2}(\mathbb{R})$.

Proof. By the Fundamental Property of Bases, it suffices to prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ spans $M_{2\times 2}(\mathbb{R})$ and is linearly independent.

Spanning. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.

Linear Independence. Suppose $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4 = \vec{0}$. Then

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

whence $\alpha_i = 0$ for all i. □

7.4 Recall that a magic square is a square array of numbers such that each row, each column, and the two main diagonals have the same sum. In class we saw two examples of 3×3 magic squares:

1	1	1
1	1	1
1	1	1

8	1	6
3	5	7
4	9	2

(a) The square on the above right uses each of the numbers from 1 to 9 exactly once. Determine all 3×3 magic squares with this property. Prove that you've found all of them. [*Hint: what can you say about the central square?*]

Consider an arbitrary magic square of the desired form:

a	b	c
d	e	f
g	h	i

so that $\{a, b, \ldots, i\} = \{1, 2, \ldots, 9\}$.

Lemma 1. *The magic sum (i.e. the sum of any row, column, or main diagonal) is 15.*

Proof. Summing all the entries of the magic square gives

$$a + b + \cdots + i = 1 + \cdots + 9 = 45.$$

Interpreting the sum on the left side as the sum of the three rows – which we know all have the same sum – we deduce that each row sums to 15. It follows from the definition of a magic square that every column and the two main diagonals also sum to 15. □

Lemma 2. *The center square must be 5.*

Proof. Summing the diagonals, the middle row and middle column, we get:

$$\begin{aligned} 60 &= (a + e + i) + (c + e + g) + (b + e + h) + (d + e + f) \\ &= a + b + \cdots + i + 3e \\ &= 45 + 3e \\ \Rightarrow e &= 5 \end{aligned}$$

The claim follows. □

We started the problem with nine variables; we now know the magic sum and the central square, which allows us to cut down on the number of variables considerably. Writing the top left corner in the form $5 + x$ and the top right corner in the form $5 + y$ allows us to fill in the rest of the magic square:

$5 + x$	$5 - x - y$	$5 + y$
$5 - x + y$	5	$5 + x - y$
$5 - y$	$5 + x + y$	$5 - x$

(Note that our choice of normalization – writing $5 + x$ rather than x – makes the entries more symmetric. The reason for this normalization is that the average entry should be 5, so we write everything relative to this average.) We may assume that both top left and right corners are larger than 5, since otherwise we could rotate the magic square until this were true. Phrased differently, we are assuming that x and y are both positive. We may further assume that $x > y$ (else reflect across the vertical axis). *cont'd...*

Given the above assumptions, the smallest quantity appearing in the magic square is clearly $5 - x - y$, hence this must equal 1. This implies that $x + y = 4$. Since $x > y$, this means that $x = 3$ and $y = 1$. Thus, up to reflection across the vertical axis combined with rotations, we have shown that the only possible magic square is

8	1	6
3	5	7
4	9	2

Rotating and reflecting produces seven other squares. Here is a collection of all eight possible magic squares:

8	1	6
3	5	7
4	9	2

4	3	8
9	5	1
2	7	6

2	9	4
7	5	3
6	1	8

6	7	2
1	5	9
8	3	4

6	1	8
7	5	3
2	9	4

2	7	6
9	5	1
4	3	8

4	9	2
3	5	7
8	1	6

8	3	4
1	5	9
6	7	2

(b) Let $MSS_n(\mathbb{R})$ denote the vector space of $n \times n$ magic squares with real entries. What is the dimension of $MSS_2(\mathbb{R})$? Prove it.

Consider a 2×2 magic square

a	b
c	d

In order for this to indeed be a magic square, we must have $a + b = a + c$, and so $b = c$. Similarly $a + b = b + d$, and so $a = d$. Finally $a + b = a + d$ implies $b = d$, at which point we can conclude that every 2×2 magic square has the form

$$\begin{array}{|c|c|}\hline \alpha & \alpha \\ \hline \alpha & \alpha \\ \hline \end{array} = \alpha \cdot \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$$

Thus the set $\left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \right\}$ is a basis for $MSS_2(\mathbb{R})$, whence $\dim MSS_2(\mathbb{R}) = 1$.

(c) What is the dimension of $MSS_3(\mathbb{R})$? Prove it.

We call the sum of any row of the magic square the *magic sum* of the square. The first observation is similar to what we saw in part (a):

Lemma 3. *The magic sum of a magic square is three times its central entry.*

Proof. Given a 3×3 magic square, let M denote the magic sum and x denote the central entry. Then the sum over all entries in the square is $3M$ (since each of the three rows sums to M). Summing the middle row, the middle column, and the two diagonals yields

$$3x + 3M = 4M,$$

whence $M = 3x$ as claimed. $\square$

cont'd...

Denoting the central entry by x, the top left corner entry by $x+y$, and the top right corner entry by $x+z$, and applying the Lemma, we can fill in the rest of the magic square:

$$\begin{array}{|c|c|c|}\hline x+y & x-y-z & x+z \\ \hline x-y+z & x & x+y-z \\ \hline x-z & x+y+z & x-y \\ \hline \end{array} =$$

$$= x \cdot \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} + y \cdot \begin{array}{|c|c|c|}\hline 1 & -1 & 0 \\ \hline -1 & 0 & 1 \\ \hline 0 & 1 & -1 \\ \hline \end{array} + z \cdot \begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline 1 & 0 & -1 \\ \hline -1 & 1 & 0 \\ \hline \end{array}$$

This gives us a basis of $MSS_3(\mathbb{R})$:

$$\left\{ \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & -1 & 0 \\ \hline -1 & 0 & 1 \\ \hline 0 & 1 & -1 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline 1 & 0 & -1 \\ \hline -1 & 1 & 0 \\ \hline \end{array} \right\}$$

We conclude that $MSS_3(\mathbb{R})$ has dimension 3.