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MATH 250 : LINEAR ALGEBRA

Problem Set 8 – KEY

8.1 In class we sketched a proof that any linearly independent set in a finite-dimensional vector space is contained in a basis of that space. The goal of this exercise is to complete this proof. Throughout, let V be a finite-dimensional vector space, and recall that for any finite set $A = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq V$ we define

$$\text{span } A := \left\{\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n : \alpha_j \in \mathbb{R}\ \forall j\right\}$$

(i.e. span A is the set of all vectors which can be formed by linearly combining the elements of A).

(a) Suppose $\mathcal{L} \subseteq V$ is linearly independent, and that $\mathcal{S} \subseteq V$ spans V. Prove that if span $\mathcal{L} \supseteq \mathcal{S}$ then $\mathcal{L}$ is a basis of V.

Since $\mathcal{L}$ is linearly independent, it suffices (by the Fundamental Property of Bases) to show that $\mathcal{L}$ spans V, i.e., that an arbitrary $\vec{v} \in V$ can be written as a linear combination of the elements in $\mathcal{L}$. Pick $\vec{v} \in V$. Since $\mathcal{S}$ spans V, we can write $\vec{v}$ as a linear combination of the elements of $\mathcal{S}$. Since $\mathcal{S} \subseteq$ span $\mathcal{L}$, every element of $\mathcal{S}$ can be written a linear combination of elements of $\mathcal{L}$. It follows that $\vec{v}$ can be written as a linear combination of elements of $\mathcal{L}$. Since $\vec{v}$ was arbitrary, we conclude that $\mathcal{L}$ spans V, hence is a basis. $\square$

(b) Suppose $\mathcal{L} \subseteq V$ is linearly independent, and that $\exists \vec{v} \in V$ such that $\vec{v} \notin$ span $\mathcal{L}$. Prove that $\mathcal{L} \cup \{\vec{v}\}$ is linearly independent.

Since V is finite-dimensional, there exists a finite spanning set. We proved in class that every spanning set is at least as large as every linearly independent set, hence $\mathcal{L}$ must be finite. Write

$$\mathcal{L} := \{\vec{\ell}_1, \vec{\ell}_2, \ldots, \vec{\ell}_n\}$$

Suppose

$$\alpha_1\vec{\ell}_1 + \cdots \alpha_n\vec{\ell}_n + \beta\vec{v} = \vec{0}$$

We immediately deduce that $\beta = 0$, else we would be able to express $\vec{v}$ as a linear combination of the $\vec{\ell}_i$'s, contradicting our hypothesis that $\vec{v} \notin$ span $\mathcal{L}$. Thus

$$\alpha_1\vec{\ell}_1 + \cdots \alpha_n\vec{\ell}_n = \vec{0}$$

But since these vectors are linearly independent, we deduce that $\alpha_i = 0$ for every i. We've therefore shown that only the trivial linear combination of elements of $\mathcal{L} \cup \{\vec{v}\}$ produces $\vec{0}$. In other words, $\mathcal{L} \cup \{\vec{v}\}$ is linearly independent! $\square$

(c) Write out a careful proof that any linearly independent set in a finite-dimensional vector space is contained in a basis of that space.

By definition, since V is finite-dimensional there must exist a finite set $\mathcal{S}$ which spans V. Let $\mathcal{L}$ be a linearly independent subset of V. There are two possibilities:

(i) span $\mathcal{L} \supseteq \mathcal{S}$, or

(ii) $\exists \vec{s}_0 \in \mathcal{S}$ such that $\vec{s}_0 \notin$ span $\mathcal{L}$.

In the former scenario, part (a) implies that $\mathcal{L}$ is a basis, hence (in particular) is contained in a basis. In the latter scenario, part (b) shows that $\mathcal{L}_1 := \mathcal{L} \cup \{\vec{s}_0\}$ is linearly independent. Now iterate the process: if span $\mathcal{L}_1 \supseteq \mathcal{S}$ then we're done, else there exists some element $\vec{s}_1 \in \mathcal{S}$ which doesn't belong to span $\mathcal{L}_1$ and we can create a new linearly independent set $\mathcal{L}_2 := \mathcal{L}_1 \cup \{\vec{s}_1\}$. This process must terminate, since $\mathcal{S}$ is finite so eventually (in the worst-case scenario) we would arrive at a set $\mathcal{L}_k$ which contains all of $\mathcal{S}$. □

8.2 Consider the set $\mathcal{F}$ of all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the differential equation $f'' + f = 0$. (Here f'' means the second derivative of f. Note that we are implicitly assuming that both f' and f'' exist, since otherwise it would be difficult to satisfy the given differential equation!)

(a) Prove that the $\mathcal{F}$ is a vector space.

First, we must identify the two operations with respect to which $\mathcal{F}$ is a vector space. Given two functions $f, g \in \mathcal{F}$, we define a new function $f + g$ by setting $(f + g)(x) := f(x) + g(x)$ for every $x \in \mathbb{R}$. Given a function $f \in \mathcal{F}$ and a real number $\alpha \in \mathbb{R}$, we define a new function $\alpha \cdot f$ by setting $(\alpha \cdot f)(x) := \alpha \cdot f(x)$ for all $x \in \mathbb{R}$.

With these notions in place, we can check the vector space axioms one at a time.

(1) **Closure.** Given $f, g \in \mathcal{F}$ and $\alpha \in \mathbb{R}$. Then

$$(f + g)'' + (f + g) = f'' + f + g'' + g = 0$$

whence $f + g \in \mathcal{F}$, and

$$(\alpha \cdot f)'' + (\alpha \cdot f) = \alpha \cdot f'' + \alpha \cdot f = \alpha \cdot (f'' + f) = 0$$

whence $\alpha \cdot f \in \mathcal{F}$.

(2) **Commutativity of addition.** Inherited from $\mathbb{R}$.

(3) **Associativity of addition.** Inherited from $\mathbb{R}$.

(4) **Existence of additive identity.** Consider the function $z : \mathbb{R} \to \mathbb{R}$ defined $z(x) = 0$ for all $x \in \mathbb{R}$. Then $z + f = f$ for all $f \in \mathcal{F}$.

(5) **Existence of additive inverses.** Given $f \in \mathcal{F}$, I claim $-f \in \mathcal{F}$ as well. This is true by closure, since $-f = -1 \cdot f$.

(6) **1 is a multiplicative identity.** Inherited from $\mathbb{R}$.

(7) **Associativity of multiplication.** Inherited from $\mathbb{R}$.

(8) **Distributivity.** Inherited from $\mathbb{R}$.

Thus $\mathcal{F}$ is a vector space. □

(b) What is the dimension of $\mathcal{F}$? Prove it! [*Hint: Differentiate the functions* $g(x) = f(x)\cos x - f'(x)\sin x$ *and* $h(x) = f(x)\sin x + f'(x)\cos x$.]

Claim. *The set* $\{\sin x, \cos x\}$ *is a basis of* $\mathcal{F}$*, whence* $\dim \mathcal{F} = 2$.

Proof. It's easy to see that $\sin x, \cos x \in \mathcal{F}$. To prove these two functions form a basis, we use the Fundamental Property of Bases: we prove they span $\mathcal{F}$, and that they're linearly independent.

Spanning. Pick $f \in \mathcal{F}$, and consider the functions g and h given in the hint. Differentiating $g(x)$ yields

$$g'(x) = f'(x)\cos x - f(x)\sin x - f''(x)\sin x - f'(x)\cos x = 0$$

and it follows that g must be a constant function; say $g(x) = a$. Similarly, we find that h is constant, say $h(x) = b$. Thus, we get the simultaneous equations

$$f(x)\cos x - f'(x)\sin x = a$$

$$f(x)\sin x + f'(x)\cos x = b$$

Multiplying the top equation by $\cos x$, the bottom equation by $\sin x$, and summing the two, we find

$$f(x) = a\cos x + b\sin x.$$

Thus, every $f \in \mathcal{F}$ can be represented as a linear combination of $\sin x$ and $\cos x$.

Linear independence. Suppose $a\sin x + b\cos x = 0$. Plugging in $x = 0$ shows that $b = 0$; plugging in $x = \frac{\pi}{2}$ shows that $a = 0$. Hence only the trivial linear combination of $\sin x$ and $\cos x$ yields the zero function. □

(c) Consider the function $T : \mathcal{F} \to \mathbb{R}^2$ defined by

$$T(f) := \big(f(0), f(\pi/2)\big)$$

Is T a linear map? Either way, justify your answer.

From above, we see that given any $f \in \mathcal{F}$ there exist unique $a, b \in \mathbb{R}$ such that

$$f(x) = a\cos x + b\sin x.$$

In terms of these coefficients, we have

$$T(f) = (a, b).$$

It's now a simple matter to check that $T(f + g) = Tf + Tg$ and $T(c \cdot f) = c \cdot Tf$ for any $f, g \in \mathcal{F}$ and any $c \in \mathbb{R}$.

Note: It's also not hard to prove that T is an isomorphism, whence $\mathcal{F}$ is isomorphic to $\mathbb{R}^2$.

8.3 Given V a finite-dimensional vector space, let $\widehat{V}$ denote the set of all linear maps $T : V \to \mathbb{R}$.

(a) Prove that $\widehat{V}$ is a vector space.

[Note: $\widehat{V}$ is called the *dual space* of V.] We first define what the operations are. Given $S, T \in \widehat{V}$, we define a new function $S + T : V \to \mathbb{R}$ by

$$(S + T)(\vec{v}) := S(\vec{v}) + T(\vec{v}).$$

Similarly, given $\alpha \in \mathbb{R}$ we define a function $\alpha \cdot T : V \to \mathbb{R}$ by

$$(\alpha \cdot T)(\vec{v}) := \alpha \cdot T(\vec{v}).$$

Next we verify the vector space axioms.

(1) **Closure.** Given $S, T \in \widehat{V}$ and $\vec{v}, \vec{w} \in V$. Then

$$(S+T)(\vec{v}+\vec{w}) = S(\vec{v}+\vec{w})+T(\vec{v}+\vec{w}) = S\vec{v}+S\vec{w}+T\vec{v}+T\vec{w} = (S+T)(\vec{v})+(S+T)(\vec{w}),$$

so $S + T$ is additive. Also

$$(S + T)(\alpha\vec{v}) = S(\alpha\vec{v}) + T(\alpha\vec{v}) = \alpha S\vec{v} + \alpha T\vec{v} = \alpha \cdot (S + T)(\vec{v})$$

whence $S + T$ scales. Thus $S + T$ is linear, and hence belongs to $\widehat{V}$. Similarly, for any $\alpha \in \mathbb{R}$ one can check that $\alpha \cdot T \in \widehat{V}$.

(2) **Commutativity of addition.** Inherited from $\mathbb{R}$.

(3) **Associativity of addition.** Inherited from $\mathbb{R}$.

(4) **Existence of additive identity.** Consider the function $z : V \to \mathbb{R}$ defined $z(\vec{v}) = 0$ for all $\vec{v} \in V$. Then $z + T = T$ for all $T \in \widehat{V}$.

(5) **Existence of additive inverses.** Given $T \in \widehat{V}$, the function $-1 \cdot T$ is an additive inverse of T.

(6) **1 is a multiplicative identity.** Inherited from $\mathbb{R}$.

(7) **Associativity of multiplication.** Inherited from $\mathbb{R}$.

(8) **Distributivity.** Inherited from $\mathbb{R}$.

(b) Prove that $\dim V = \dim \widehat{V}$.

Since V is finite-dimensional, we know from class that there exists a finite basis of V, say,

$$\{\vec{v}_1, \ldots, \vec{v}_n\}.$$

Thus $n = \dim V$.

We now construct n linear maps $T_1, T_2, \ldots, T_n \in \widehat{V}$, which we hope will form a basis of $\widehat{V}$. For each $i \in \{1, 2, \ldots, n\}$, consider the function $T_i : V \to \mathbb{R}$ defined by

$$T_i(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots \alpha_n\vec{v}_n) := \alpha_i.$$

It is straightforward to verify that T is a linear map, and that

$$T_i(\vec{v}_j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Claim. *The set $\{T_1, T_2, \ldots, T_n\}$ is a basis of $\widehat{V}$. (It follows that $\dim \widehat{V} = n = \dim V$.)*

Proof. As usual, we use the Fundamental Property of Bases.

Spanning. Given an arbitrary $T \in \widehat{V}$, set

$$\beta_i := T(\vec{v}_i).$$

Then the functions T and $\beta_1 T_1 + \beta_2 T_2 + \cdots + \beta_n T_n$ agree on every basis element $\vec{v}_i$, hence must agree everywhere:

$$T = \beta_1 T_1 + \beta_2 T_2 + \cdots + \beta_n T_n.$$

Thus the T_i's span $\widehat{V}$.

Linear independence. Suppose $\gamma_1 T_1 + \gamma_2 T_2 + \cdots + \gamma_n T_n = 0$. Then for any i we have

$$\gamma_i = (\gamma_1 T_1 + \gamma_2 T_2 + \cdots + \gamma_n T_n)(\vec{v}_i) = 0$$

which shows that only the trivial combination of the T_i's produces 0. Thus, the T_i's must be linearly independent. □

(c) Give an explicit example of a non-constant linear map $\varphi : V \to \widehat{\widehat{V}}$. (Here $\widehat{\widehat{V}}$ denotes the set of all linear maps $\widehat{V} \to \mathbb{R}$.)

Given $\vec{v} \in V$, first consider the function $\varphi_{\vec{v}} : \widehat{V} \to \mathbb{R}$ defined by

$$\varphi_{\vec{v}}(T) := T(\vec{v}).$$

(This is called the *evaluation map*.) Note that $\varphi_{\vec{v}}$ is a linear map, since

$$\varphi_{\vec{v}}(S + T) = (S + T)(\vec{v}) = S\vec{v} + T\vec{v} = \varphi_{\vec{v}}(S) + \varphi_{\vec{v}}(T)$$

and

$$\varphi_{\vec{v}}(\alpha \cdot T) = (\alpha \cdot T)(\vec{v}) = \alpha \cdot (T\vec{v}) = \alpha \cdot \varphi_{\vec{v}}(T).$$

Now define $\varphi : V \to \widehat{\widehat{V}}$ by $\varphi(\vec{v}) := \varphi_{\vec{v}}$. It remains only to prove that φ is linear. We have

$$\varphi(\vec{v} + \vec{w}) = \varphi(\vec{v}) + \varphi(\vec{w})$$

since

$$\varphi_{\vec{v}+\vec{w}}(T) = T(\vec{v} + \vec{w}) = T\vec{v} + T\vec{w} = \varphi_{\vec{v}}(T) + \varphi_{\vec{w}}(T).$$

Similarly,

$$\varphi(\alpha\vec{v}) = \alpha\varphi(\vec{v})$$

since

$$\varphi_{\alpha\vec{v}}(T) = T(\alpha\vec{v}) = \alpha T\vec{v} = \alpha\varphi_{\vec{v}}(T).$$