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MATH 250 : LINEAR ALGEBRA

Problem Set 9 – KEY

9.1 Suppose V and W are vector spaces. A linear map $T : V \to W$ is called *left-invertible* iff there exists a linear map $L : W \to V$ such that $L \circ T = I_V$, and is called *right-invertible* if and only if there exists a linear map $R : W \to V$ such that $T \circ R = I_W$. (Here I_V denotes the identity map on V, i.e. $I_V(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$.) Prove that a linear map $T : V \to W$ is invertible (according to our definition from class) if and only if T is both left- and right-invertible.

($\Rightarrow$) We proved in class that if T is invertible, then $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$. Thus T^{-1} is both a right and a left inverse of T. In particular, T is left- and right-invertible.

($\Leftarrow$) Let L and R be left and right inverses of T, respectively. Pick $\vec{w} \in W$. By definition, $TR(\vec{w}) = \vec{w}$, whence $R(\vec{w}) \in T^{-1}(\vec{w})$. In particular, $\boxed{\#T^{-1}(\vec{w}) \geq 1.}$ On the other hand, suppose $\vec{v}_1$ and $\vec{v}_2 \in T^{-1}(\vec{w})$. Then $T(\vec{v}_1) = T(\vec{v}_2)$, whence $\vec{v}_1 = LT(\vec{v}_1) = LT(\vec{v}_2) = \vec{v}_2$. This shows $\boxed{\#T^{-1}(\vec{w}) \leq 1.}$ Combining the boxed inequalities, we see that $\#T^{-1}(\vec{w}) = 1$. Since $\vec{w} \in W$ was arbitrary, we conclude that T is invertible.

9.2 If V is isomorphic to W, we write $V \simeq W$. Prove that $\simeq$ is an equivalence relation.

We must show that the relation $\simeq$ satisfies the three properties of equivalence relations.

- Reflexivity. Given a vector space V, the identity map I_V defined in part (a) is an isomorphism from V to V.
- Symmetry. Suppose $V \simeq W$, and let $T : V \to W$ be an isomorphism. Then Proposition 3 of Lecture 31 implies that $T^{-1} : W \to V$ is also an isomorphism, whence $W \simeq V$.
- Transitivity. Suppose $X \simeq V$ and $V \simeq W$; say $T : X \to V$ and $S : V \to W$ are isomorphisms. I claim that $ST : X \to W$ is an isomorphism. Indeed, pick $\vec{w} \in W$. Then there exists $\vec{v} \in V$ such that $S\vec{v} = \vec{w}$, and there exists a $\vec{x} \in X$ such that $T\vec{x} = \vec{v}$; it follows that $ST\vec{x} = \vec{w}$, so $\boxed{\#(ST)^{-1}(\vec{w}) \geq 1.}$ Next, suppose $ST(\vec{x}_1) = ST(\vec{x}_2) = \vec{w}$. Since S is invertible, we deduce that $T\vec{x}_1 = T\vec{x}_2$; since T is invertible, we conclude that $\vec{x}_1 = \vec{x}_2$. This shows that $\boxed{\#(ST)^{-1}(\vec{w}) \leq 1.}$ Combining the boxed quantities, we see that ST is invertible, hence an isomorphism. Therefore, $X \simeq W$.

9.3 Find an example of a finite-dimensional vector space V and a subset $W \subseteq V$ such that W is a vector space, but is *not* a subspace of V.

There are many possible solutions to this, but all of them have one feature in common: the notions of 'addition' and 'scalar multiplication' must be different for W than the ones for V. The moral of this problem is: *a vector space is not simply a set, but a set along with two operations.*

Here's an example. Take V to be $\mathbb{R}$ under the usual notions of $+$ and $\cdot$. Now let $W := \mathbb{R}_{>0}$ (the set of all positive real numbers); clearly $W \subseteq V$. Now we define the following two operations on W:

- $x \oplus y := xy$
- $\alpha \cdot x := x^\alpha$

It is an exercise to check that W forms a vector space with respect to these operations. As an example, let's check one of the distributive laws:

$$\alpha \cdot (x \oplus y) = (xy)^\alpha = x^\alpha y^\alpha = (\alpha \cdot x) \oplus (\alpha \cdot y).$$

Thus, W is a subset of V, is a vector space with respect to some operations, but is *not* a subspace of V.

9.4 Given V a finite-dimensional vector space.

(a) Suppose W is a subspace of V. Prove that $W = V$ iff $\dim W = \dim V$. [*Hint: Use problem **8.1***]

If $W = V$ then of course $\dim W = \dim V$, so it suffices to prove the converse. Thus, suppose $\dim W = \dim V$. Let $\{\vec{w}_1, \dots, \vec{w}_n\}$ be a basis of W; in particular, these vectors are linearly independent. By problem **8.1** we know that they must be contained in a basis of V. On the other hand, any basis of V must also have n vectors in it by hypothesis. Thus, $\{\vec{w}_1, \dots, \vec{w}_n\}$ must already be a basis of V. This shows that any vector in V can be written as a linear combination of the $\vec{w}_i$'s, hence is contained in W. On the other hand, every vector in W is clearly contained in V. It follows that $W = V$ as claimed. $\square$

(b) Suppose $T : V \to V$ is a linear map. Prove that T is an isomorphism if and only if $\ker T = \{\vec{0}\}$.

Once again, the forward direction is straightforward: if T is an isomorphism, then it is invertible, whence $T^{-1}(\vec{0}) = \vec{0}$ as claimed. Thus it suffices to prove the converse. Suppose $\ker T = \{\vec{0}\}$. The rank-nullity theorem asserts that

$$\dim(\text{im } T) + \dim(\ker T) = \dim V.$$

Since $\ker T$ has dimension 0, it follows that

$$\dim(\text{im } T) = \dim V.$$

Since $\text{im } T$ is a subspace of V, part (a) implies that $\text{im } T = V$.

Now we are ready to show that T is an isomorphism. Pick any $\vec{w} \in V$. Since $\text{im } T = V$, we have $\boxed{\#T^{-1}\vec{w} \geq 1.}$ Now suppose $T\vec{v}_1 = \vec{w}$ and $T\vec{v}_2 = \vec{w}$. Then in particular $T\vec{v}_1 = T\vec{v}_2$, whence (by additivity) $T(\vec{v}_1 - \vec{v}_2) = \vec{0}$. But $\ker T = \{\vec{0}\}$, so $\vec{v}_1 - \vec{v}_2 = \vec{0}$! This shows that any two elements in the preimage of $\vec{w}$ must be the same, i.e. that $\boxed{\#T^{-1}\vec{w} \leq 1.}$ Combining the two boxed results shows that T is invertible, hence an isomorphism. $\square$