MAT 302: LECTURE SUMMARY

In this lecture we will give some intuition for Stirling's formula, which we will prove rigorously in the following lecture. Although Stirling's formula is useful in its own right, our main purpose in proving it is to introduce and use some notations and tools from analysis (rates of growth, big-Oh notation, partial summation, the Euler-Maclaurin formula, etc).

Our goal is to say something about the size of $N!$. What can we say? A lower bound was immediately suggested:

$$\begin{aligned} N! &= N \times (N-1) \times \cdots \times 2 \times 1 \\ &\geq 2 \times 2 \times \cdots \times 2 \times 1 \\ &= 2^{N-1} \end{aligned}$$

A similar idea gives an upper bound:

$$N! \leq N^N.$$

So, the true size of $N!$ lies somewhere between 2^{N-1} and N^N. Stirling's formula gives the answer (for large values of N):

Stirling's Formula. $N! \sim \sqrt{2\pi N} \left(\dfrac{N}{e}\right)^N$

The notation $\sim$ (read: 'is asymptotic to') means that the ratio of the two functions tends to 1. In other words,

$$f(x) \sim g(x) \quad \iff \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Note that two functions might be quite different, and yet still be asymptotic to one another. For example, $x + 1000\sqrt{x} \sim x$.

How do our upper and lower bounds on $N!$ compare to Stirling's formula? N^N might look like a rather promising estimate. However, as N gets large, $\sqrt{2\pi N}\left(\frac{N}{e}\right)^N$ becomes smaller and smaller relative to N^N. More precisely, the ratio of these two tends to 0 as N tends to infinity. There is a convenient notation for this: we say that $f(x) = o\big(g(x)\big)$ if $\frac{f(x)}{g(x)} \longrightarrow 0$ as $x \to \infty$. Intuitively, this just means that $f(x)$ is eventually tiny compared to $g(x)$.

So, our estimates are pretty far from the truth. But how can we do better? The trick is to study not $N!$, but $\log N!$. (In this course, $\log$ will always refer to the natural logarithm.) Recall that $\log ab = \log a + \log b$, so that

(*)
$$\log N! = \sum_{n \leq N} \log n.$$

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Why does this help? *Because sums can be approximated by integrals.* For example, consider the graph of $\log x$. We use the classical Riemann approximation to the integral of this by putting down a rectangle with base between $x = 1$ and $x = 2$ of height $\log 2$, a second rectangle with base between $x = 2$ and $x = 3$ of height $\log 3$, etc. The sum of these rectangles (an example of a right Riemann sum) is precisely the sum in (*), and is a fair approximation to the integral

$$\int_1^N \log t \, dt.$$

The integral is easily evaluated, leading us to conclude that

$$\log N! \approx N \log N - N + 1.$$

This in turn implies that $N! \approx e \times \left(\frac{N}{e}\right)^N$, which is certainly starting to look a lot like Stirling's formula. Unfortunately, this is still quite far from the truth! You should check that

$$e \times \left(\frac{N}{e}\right)^N = o\left(\sqrt{2\pi N}\left(\frac{N}{e}\right)^N\right).$$

So if we hope to prove Stirling's formula, we'll have to find a better estimate.

There are much better approximations to $\displaystyle\int_1^N \log t \, dt$ than that given by rectangles of width 1. For example, we could better approximate the shape of the curve by replacing the rectangles in our above picture by trapezoids, still having the same base as before but now having its two other vertices lying on the curve $\log x$. Since the area of a trapezoid is its height times the average of the bases, we see that

$$\begin{aligned}\int_1^N \log t \, dt &\approx \frac{1}{2}(\log 1 + \log 2) + \frac{1}{2}(\log 2 + \log 3) + \frac{1}{2}(\log 3 + \log 4) + \cdots + \frac{1}{2}(\log(N-1) + \log N) \\ &= \sum_{n \le N} \log n - \frac{1}{2}\log 1 - \frac{1}{2}\log N\end{aligned}$$

It follows that

$$\log N! \approx \int_1^N \log t \, dt + \frac{1}{2}\log 1 + \frac{1}{2}\log N$$

from which we deduce the approximation

$$N! \approx e\sqrt{N}\left(\frac{N}{e}\right)^N.$$

Although this is not Stirling's formula, it's the first time that we're not totally off: the right hand side is neither tiny nor huge compared to the true value, but is merely a constant away.

Note that to get this estimate we used the 'trapezoid rule', which you were taught in your calculus course as a trick to approximate integrals. In our scenario, we know the exact value of the integral in question, so reverse the process and use it to approximate the sum.

Next lecture, we will make all this precise, by introducing a new tool: partial summation.