MAT 302: LECTURE SUMMARY

In the previous lecture, we discussed an approach to approximating the size of $N!$. Rather than looking at $N!$ directly, we considered $\log N!$; the advantage is that

$$\log N! = \sum_{n \leq N} \log n$$

which can be approximated in terms of the integral of $\log t$ as t ranges from 1 to N. In particular, we saw that using the 'trapezoid rule' leads to a decent approximation. This approach flips the usual trapezoid rule on its head – rather than approximating an integral by a sum, we approximate a sum by an integral.

This lecture, we will make this precise by introducing the technique of partial summation. This can be stated as follows:

Partial Summation. *For any sequence $a_1, a_2, \ldots$ and any nice function $f(x)$, we can write*

$$\sum_{n \leq N} a_n f(n) = \int_{1^-}^{N} f(t)\, d\Big(\sum_{n \leq t} a_n\Big) \tag{†}$$

The first order of business is to explain the notation on the right hand side. The bounds in the integral mean that t runs continuously from slightly below 1 to N. When $t < 1$, the sum

$$\sum_{n \leq t} a_n \tag{*}$$

is 0 (since there are no positive integers below 1), so all such t do not contribute anything to the integral. As t crosses the value 1, the sum (*) suddenly changes value, becoming a_1. Since the change in the sum is a_1, and this happens when t crosses 1, the integral in (†) picks up $f(1)a_1$. Then, as t moves from slightly above 1 to slightly below 2, the sum (*) doesn't change (it equals a_1 the whole time), so none of the t in this range contribute to the integral. As t crosses 2, the sum (*) suddenly increases by a_2, so the integral picks up $f(2)a_2$. This process continues, and by the end the integral has picked up $a_1 f(1) + a_2 f(2) + \cdots + a_N f(N)$. This justifies the notation in (†).

So far, we haven't yet done anything other than rewrite a perfectly innocent sum in a weird way, using an integral sign. In fact, if you stare at the integral in (†), you may realize that you've dealt with similar integrals many times; recall the integration by parts formula

$$\int u\, dv = uv - \int v\, du.$$

To evaluate our integral, we will simply take $u = f(t)$ and $v = \sum_{n \leq t} a_n$ and integrate by parts.

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You have probably seen in previous courses that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. (You should take a moment to try to prove this!) From a computational standpoint, this is somewhat surprising. The sum of the first 1000 terms of the series is about 7.5; one has to add 800 more terms to get the sum up to 8; even if one adds 10,000 more terms, the sum remains less than 10. And yet, it keeps growing. How quickly does it grow? Is there an easier way to approximate the sum of the first 10,000,000,000 terms, other than just adding them up?

We will use partial summation to do exactly that. We start by writing

$$\sum_{n \le N} \frac{1}{n} = \int_{1^-}^{N} \frac{1}{t}\, d\Big(\sum_{n \le t} 1\Big) = \int_{1^-}^{N} \frac{1}{t}\, d[t]$$

where $[t]$ (read: 'the floor of t') denotes the largest integer $\le t$. Integrating by parts, we find

$$\begin{aligned} \int_{1^-}^{N} \frac{1}{t}\, d[t] &= \frac{1}{t}\,[t]\Big|_{1^-}^{N} + \int_{1^-}^{N} \frac{[t]}{t^2}\, dt \\ &= 1 + \int_{1}^{N} \frac{[t]}{t^2}\, dt \end{aligned}$$

Note that in the first term, we used that $[1^-] = 0$ (since 1^- is slightly smaller than 1). In the integral, on the other hand, we can replace 1^- by 1, simply by letting the lower bound of the integral tend to 1 from below.

Note that for any t, we can write $t = [t] + \{t\}$, where $\{t\}$ is the fractional part of t. (For example, $\pi = 3 + 0.141592\ldots$.) Therefore, we can rewrite the integral from above in the form

$$\begin{aligned} \int_{1}^{N} \frac{[t]}{t^2}\, dt &= \int_{1}^{N} \frac{t - \{t\}}{t^2}\, dt \\ &= \int_{1}^{N} \frac{dt}{t} - \int_{1}^{N} \frac{\{t\}}{t^2}\, dt \\ &= \log N - \int_{1}^{N} \frac{\{t\}}{t^2}\, dt \end{aligned}$$

How big is this last integral? Well, since $0 \le \{t\} < 1$, we see that

$$0 \le \int_{1}^{N} \frac{\{t\}}{t^2}\, dt < \int_{1}^{N} \frac{dt}{t^2} = 1 - \frac{1}{N} \le 1.$$

Putting together everything we've done so far, we obtain

$$\sum_{n \le N} \frac{1}{n} = \log N + \Big(\text{something between 0 and 1}\Big).$$

Since it is inconvenient to write this out in words every time, we will use the shorthand

$$\sum_{n \le N} \frac{1}{n} = \log N + O(1).$$

Formally, we shall say that $f(x) = O\big(g(x)\big)$ (read: 'f is big-oh of g') if there exists some positive constant C such that $\left|\frac{f(x)}{g(x)}\right| \le C$ for all x for which $\frac{f(x)}{g(x)}$ is defined. Since we are only interested in the magnitude of $g(x)$, when using this notation we will always take $g(x)$ to be a positive function. Note that C is an absolute constant – we're not allowed to change it based on x!

Big-oh is a very convenient shorthand when you don't care about an exact formula for a function, but are only interested in its size. For example the equation written above, $\sum \frac{1}{n} = \log N + O(1)$, is shorthand for $\sum \frac{1}{n} = \log N + E(N)$, where $E(N) = O(1)$.

AN EXAMPLE OF BIG OH NOTATION

To get a better sense of how this notation works, let's consider an example:

$$\begin{aligned} x - 1000\sqrt{23x+40} + 10\sin x &= x - 1000\sqrt{23x+40} + O(1) \\ &= x + O(\sqrt{x}) \end{aligned}$$

We analyze this step by step.

(1) $|10 \sin x| \le 10$, so we can replace this term by $O(10)$. Why do we write $O(1)$ instead? Because the big-oh notation forgets about the implicit constant; $O(10)$ and $O(1)$ are indistinguishable, and $O(1)$ is simpler to write!

(2) It is straightforward to see that $1000\sqrt{23x+40} = O(\sqrt{23x+40})$. But then why do we write $O(\sqrt{x})$? Consider the ratio

$$r(x) = \frac{\sqrt{23x+40}}{\sqrt{x}}.$$

As x gets large, $r(x)$ tends to $\sqrt{23}$. It follows that $\sqrt{23x+40} \le 5\sqrt{x}$ for *all* sufficiently large x. (Make sure you stop here and understand the previous sentence.) What about those x which are not 'sufficiently large'? Let's say sufficiently large means $x \ge 1,000,000$. For all x smaller than this, $r(x)$ is bounded by some constant – and we don't care which! The point is, wherever $r(x)$ is a nice function, it is bounded by some constant, which explains why $1000\sqrt{23x+40} = O(\sqrt{x})$.

(3) Why do we write $x + O(\sqrt{x})$, when the square-root is actually being subtracted, not added? When you use $O(\cdot)$ notation, you're indicating only the size of the error (up to a constant multiple). Because of this, whether you add or subtract isn't relevant.

A SECOND APPLICATION OF PARTIAL SUMMATION

Now we consider instead the sum $\displaystyle\sum_{n\le N}\frac{1}{n^2}$. This sum converges as $N\to\infty$ (in fact, Euler proved that it converges to $\frac{\pi^2}{6}$). We quickly explored what happens when we attack this problem using partial summation.

Following the approach from above, we find:

$$\begin{aligned}\sum_{n\le N}\frac{1}{n^2} &= \int_{1^-}^{N}\frac{1}{t^2}\,d\Big(\sum_{n\le t}1\Big)=\int_{1^-}^{N}\frac{1}{t^2}\,d[t]\\ &= \frac{[t]}{t^2}\bigg|_{1^-}^{N}+2\int_{1}^{N}\frac{[t]}{t^3}\,dt\\ &= \frac{1}{N}+2\int_{1}^{N}\frac{t-\{t\}}{t^3}\,dt\\ &= \frac{1}{N}+2\int_{1}^{N}\frac{dt}{t^2}-2\int_{1}^{N}\frac{\{t\}}{t^3}\,dt\\ &= \frac{1}{N}+2\left(1-\frac{1}{N}\right)+O(1)\\ &= O(1).\end{aligned}$$

THE EULER-MACLAURIN FORMULA

From the two applications of partial summation above, you should see a general approach emerge. We apply the same attack to the sum of a general function:

$$\begin{aligned}\sum_{n\le N}f(n) &= \int_{1^-}^{N}f(t)\,d[t]\\ &= N\,f(N)-\int_{1}^{N}[t]f'(t)\,dt\\ &= N\,f(N)-\int_{1}^{N}(t-\{t\})f'(t)\,dt\\ &= N\,f(N)-\int_{1}^{N}t\,f'(t)\,dt+\int_{1}^{N}\{t\}\,f'(t)\,dt \qquad (\ddagger)\end{aligned}$$

A little inspiration and experience helps to recognize that the first two terms of the last equation are essentially what one gets upon integrating $f(t)$ by parts:

$$\begin{aligned}\int_{1}^{N}f(t)\,dt &= t\,f(t)\bigg|_{1}^{N}-\int_{1}^{N}t\,f'(t)\,dt\\ &= N\,f(N)-f(1)-\int_{1}^{N}t\,f'(t)\,dt.\end{aligned}$$

Plugging this back into (‡) yields

$$(\clubsuit) \qquad \sum_{n\leq N} f(n) = \int_1^N f(t)\ dt + f(1) + \int_1^N \{t\}\, f'(t)\ dt$$

(This can be viewed as a rigorous form of the right Riemann sum approximation to an integral.)

This is a nice formula, but now we make it even nicer with a trick. If $f(t)$ is a monotonic function (i.e. always increasing, or always decreasing) then $f'(t)$ will be always positive or always negative. In this case, it is advantageous to integrate $f'(t)$ against $\left(\{t\} - \frac{1}{2}\right)$ rather than against $\{t\}$ as we do in (♣). The reason is simple: $\{t\}$ constantly runs from from 0 to 1, whereas $\left(\{t\} - \frac{1}{2}\right)$ runs from $-\frac{1}{2}$ to $\frac{1}{2}$, leading to more cancellation in the integral in (♣). Accordingly, we rewrite (♣) in the form

$$\sum_{n\leq N} f(n) = \int_1^N f(t)\ dt + f(1) + \int_1^N \left(\{t\} - \frac{1}{2}\right)\ f'(t)\ dt + \frac{1}{2}\int_1^N f'(t)\ dt.$$

Simplifying, we obtain the following formula:

$$\sum_{n\leq N} f(n) = \int_1^N f(t)\ dt + \frac{1}{2}f(1) + \frac{1}{2}f(N) + \int_1^N \left(\{t\} - \frac{1}{2}\right)\ f'(t)\ dt$$

This can be viewed as the rigorous version of the trapezoid rule, and is the simplest case of a more general identity known as the Euler-Maclaurin formula. To obtain the general case, one integrates the final integral on the right hand side by parts repeatedly.

Stirling's formula

We finally were in a position to approach Stirling's formula rigorously. From the Euler-Maclaurin formula, we find that

$$\begin{aligned}\sum_{n\leq N} \log n &= \int_1^N \log t\ dt + \frac{1}{2}\log N + \int_1^N \left(\{t\} - \frac{1}{2}\right)\ \frac{dt}{t} \\ &= N\log N - N + \frac{1}{2}\log N + 1 + \int_1^N \left(\{t\} - \frac{1}{2}\right)\ \frac{dt}{t}\end{aligned}$$

An application of the alternating series test shows that the integral on the right hand side converges, from which we deduce the approximation

$$\log N! = N\log N - N + \frac{1}{2}\log N + O(1)$$

Exponentiating both sides yields

$$N! = e^{O(1)}\sqrt{N}\ \left(\frac{N}{e}\right)^N$$

What does $e^{O(1)}$ mean? $O(1)$ is a quantity whose magnitude is bounded by a constant. Thus the value of $e^{O(1)}$ is bounded between two positive constants. In other words, there exist constants

$\beta \geq \alpha > 0$ such that

$$\alpha\sqrt{N}\left(\frac{N}{e}\right)^N \leq N! \leq \beta\sqrt{N}\left(\frac{N}{e}\right)^N$$

In other words, we've figured out the rate of growth of $N!$, up to a constant factor. This is pretty great! We abbreviate the above expression using yet another convenient notation:

$$N! \asymp \sqrt{N}\left(\frac{N}{e}\right)^N$$

Formally, $f(x) \asymp g(x)$ means that there exist positive constants α and β such that $\alpha\, g(x) \leq f(x) \leq \beta\, g(x)$ for all sufficiently large x.

We're now very close to proving Stirling's asymptotic formula

$$N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N .$$

All that remains is to understand the asymptotic behavior of $\int_1^N \left(\{t\} - \frac{1}{2}\right)\frac{dt}{t}$. One method of accomplishing this is outlined on the first problem set.