Last time, we proved that

$$\pi(2n) - \pi(n) \ll \frac{n}{\log n} \tag{1}$$

for every integer $n \geq 2$. We will deduce from this the following upper bound:

<u>Theorem (Chebyshev):</u> $\pi(x) \ll \frac{x}{\log x}$ for all real numbers $x \geq 2$.
<u>Proof:</u> As a first step, we prove

$$\pi(x) - \pi(\tfrac{x}{2}) \ll \frac{x}{\log x} \qquad \forall x \geq 2. \tag{2}$$

Of course, this looks an awful lot like equation (1) above, and the natural temptation is to apply it with $n = \frac{x}{2}$. The problem is that (1) only applies when n is an integer, so we can't set $n := \frac{x}{2}$. Instead, we do the next best thing: set $n := \lfloor \frac{x}{2} \rfloor$. Then $\pi(\frac{x}{2}) = \pi(n)$. What can we say about $\pi(x)$ in terms of n? Well, by definition of the floor function we have

$$n \leq x/2 < n + 1.$$

Thus $2n \leq x < 2n + 2$, whence $\pi(x) = \pi(2n)$ or $\pi(2n+1)$; in particular, we deduce

$$\pi(x) \leq \pi(2n) + 1.$$

(Why?) Thus we have

$$\pi(x) - \pi(\frac{x}{2}) \leq \pi(2n) + 1 - \pi(n) = \pi(2n) - \pi(n) + 1 \leq C\frac{n}{\log n} + 1$$

from (1). (Here C is some positive constant.) Since $1 \leq \frac{n}{\log n}$ for large enough n, we deduce that

$$\pi(x) - \pi(\frac{x}{2}) \ll \frac{n}{\log n}.$$

The final step is to rewrite the right hand side in terms of x. This is where the $\ll$ symbol becomes very useful:

$$\frac{n}{\log n} \leq \frac{x/2}{\log(x/2 - 1)} \ll \frac{x}{\log x}.$$

Intuitively, this should be clear: when x is large, subtracting 1 from $\frac{x}{2}$ doesn't really change the size by much, and $\log x/2 = \log x - \log 2$ is roughly the same size as $\log x$. To argue this more rigorously, observe that for all large x we have $\log(x/2 - 1) \geq \log x/4 \geq \log \sqrt{x} \gg \log x$. Putting all our work together, we've proved (2)!

Now we're ready for the final act: we will prove that

$$\pi(x) \ll \frac{x}{\log x}. \tag{3}$$

Alex points out that (2) implies $\pi(x) \leq \pi(x/2) + \frac{Cx}{\log x}$, so if we somehow knew that $\pi(x/2)$ were small then we'd be able to prove (3). Unfortunately, it's not clear how to show that $\pi(x/2)$ is small, since this is essentially the same as the assertion we're trying to prove!

Konnor came up with a different approach. He observed that (2) implies not just a single inequality, but many different ones:

$$\begin{aligned}
\pi(x) - \pi(x/2) &\leq C \cdot \frac{x}{\log x} \\
\pi(\frac{x}{2}) - \pi(\frac{x}{4}) &\leq C \cdot \frac{x/2}{\log x/2} \\
\pi(\frac{x}{4}) - \pi(\frac{x}{8}) &\leq C \cdot \frac{x/4}{\log x/4} \\
&\vdots \\
\pi(\frac{x}{2^\ell}) - \pi(\frac{x}{2^{\ell+1}}) &\leq C \cdot \frac{x/2^\ell}{\log x/2^\ell}
\end{aligned}$$

The absolutely crucial observation here is that *the constants C appearing on the right hand sides are all the same constant.* This is the power of the bound (2): that the implicit constant is independent of x.

The natural next step is to sum all these inequalities up. The sum of all the left hand sides is super nice: everything cancels apart from the first and last term, and if we take ℓ large enough then $\pi(\frac{x}{2^{\ell+1}})$ becomes very small and we can employ a version of Alex's idea from above. The problem now becomes: what can we say about the sum of the right hand sides? We have a bunch of fractions, each with a different denominator... it's a mess to add them up! Fortunately, there's a trick.

Pick ℓ such that $\frac{x}{2^{\ell+1}} \leq \sqrt{x} < \frac{x}{2^\ell}$. Then

$$\begin{aligned}
\pi(x) - \pi(x/2) &\leq C \cdot \frac{x}{\log x} && \leq C \cdot \frac{x}{\log \sqrt{x}} \\
\pi(\frac{x}{2}) - \pi(\frac{x}{4}) &\leq C \cdot \frac{x/2}{\log x/2} && \leq C \cdot \frac{x/2}{\log \sqrt{x}} \\
\pi(\frac{x}{4}) - \pi(\frac{x}{8}) &\leq C \cdot \frac{x/4}{\log x/4} && \leq C \cdot \frac{x/4}{\log \sqrt{x}} \\
&\vdots \\
\pi(\frac{x}{2^\ell}) - \pi(\frac{x}{2^{\ell+1}}) &\leq C \cdot \frac{x/2^\ell}{\log x/2^\ell} && \leq C \cdot \frac{x/2^\ell}{\log \sqrt{x}}
\end{aligned}$$

Now the right hand sides all have the same denominator, so we can add them up:

$$\pi(x) - \pi\left(\frac{x}{2^{\ell+1}}\right) \leq C \cdot \frac{x}{\log \sqrt{x}} \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) \leq \frac{2Cx}{\frac{1}{2}\log x} = \frac{4Cx}{\log x}.$$

Now we apply Alex's idea:

$$\pi(x) \le \frac{4Cx}{\log x} + \pi\left(\frac{x}{2^{\ell+1}}\right) \le \frac{4Cx}{\log x} + \pi\left(\sqrt{x}\right) \le \frac{4Cx}{\log x} + \sqrt{x} = \frac{x}{\log x}\big(4C + o(1)\big) \ll \frac{x}{\log x}. \qquad \square$$

This technique is called "splitting into dyadic blocks", the 'dyadic' comes from 2^k to 2^{k+1}.

<u>Theorem (Chebyshev):</u> $\pi(x) \gg \frac{x}{\log x}$
<u>Proof:</u> Consider $\binom{2n}{n}$ once more. Consider its prime factorization:

$$\binom{2n}{n} = \prod_p p^{\nu_p(\binom{2n}{n})} \implies \log\binom{2n}{n} = \sum_p \nu_p(\binom{2n}{n}) \log p = \sum_{p \le 2n} \nu_p(\binom{2n}{n}) \log p.$$

Lemma: $\nu_p(\binom{2n}{n}) \le \frac{\log 2n}{\log p}$. [We will prove this below.]
Taking the lemma on faith for now, we have

$$\log\binom{2n}{n} = \sum_{p \le 2n} \nu_p(\binom{2n}{n}) \log p \le \sum_{p \le 2n} \log 2n = \log 2n \sum_{p \le 2n} 1 = \pi(2n) \log 2n$$

which tells us that

$$\pi(2n) \ge \frac{\log\binom{2n}{n}}{\log 2n}. \tag{4}$$

Thus if we can get a nice lower bound on $\log\binom{2n}{n}$, we'll find a nice lower bound on $\pi(2n)$. Recall that

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \le \sum_{k=0}^{2n} \binom{2n}{n} = (2n+1)\binom{2n}{n}$$

whence $\binom{2n}{n} \ge \frac{2^{2n}}{2n+1}$. Taking logarithms, we find

$$\log\binom{2n}{n} \ge 2n \log 2 - \log(2n+1) = n\big(2\log 2 - o(1)\big) \gg n.$$

Plugging this into (4) gives

$$\pi(2n) \ge \frac{\log\binom{2n}{n}}{\log 2n} \gg \frac{n}{\log 2n} \gg \frac{n}{\log n}.$$

This is almost what we want; all that's left to do is to extrapolate from this a more general bound that holds for real number inputs. Just as we did before, set $n := \left\lfloor \frac{x}{2} \right\rfloor$, from which it follows that $2n \le x < 2n+2$. In particular,

$$\pi(x) \ge \pi(2n) \gg \frac{n}{\log n} \ge \frac{x/2 - 1}{\log x/2} \gg \frac{x-2}{\log x} = \frac{x\big(1 - o(1)\big)}{\log x} \gg \frac{x}{\log x}. \qquad \square$$

Actually, we're not quite finished: we need to go back and prove the lemma.

<u>Proof of Lemma:</u> First note that $\nu_p(\binom{2n}{n}) = \nu_p\big(\frac{(2n)!}{(n!)^2}\big) = \nu_p\big((2n)!\big) - 2\nu_p(n!)$. Thus we're naturally led to the question: what is $\nu_p(k!)$? To build intuition, we considered an example: what's $\nu_2(13!)$? We write out the numbers from 1 to 13 and under each number place a ✓ for each factor of 2 that number contributes:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
	✓		✓		✓		✓		✓		✓	
			✓				✓				✓	
							✓					

Counting the total number of ✓'s, we see that $\nu_2(13!) = 10$. But, staring at this diagram, we see another approach to counting the ✓'s emerge: we count by row. How many ✓'s are in the first row? There's one for each even number less than 13, and there are $\lfloor \frac{13}{2} \rfloor = 6$ of these. What about the second row? We get a second factor of 2 precisely under the multiples of 4, and there are precisely $\lfloor \frac{13}{4} \rfloor = 3$ of these. To get a third factor of 2 we need multiples of 8, and there are $\lfloor \frac{13}{8} \rfloor = 1$ of these. At this point we can stop, because there are no multiples of 16 up to 13.

This process generalizes in a straightforward way: for any p and any k we have

$$\nu_p(k!) = \sum_{j\geq 1} \left\lfloor \frac{k}{p^j} \right\rfloor = \sum_{1\leq j\leq N} \left\lfloor \frac{k}{p^j} \right\rfloor \qquad \text{for any } N \geq \frac{\log k}{\log p}.$$

Where does this endpoint come from? Well, $\lfloor k/p^j \rfloor = 0$ whenever $p^j > k$, and $p^j \leq k$ iff $j \leq \frac{\log k}{\log p}$.

Using this formula, we find

$$\begin{aligned}\nu_p(\binom{2n}{n}) &= \nu_p\big((2n)!\big) - 2\nu_p(n!)\\ &= \sum_{1\leq j\leq \frac{\log 2n}{\log p}} \left\lfloor \frac{2n}{p^j} \right\rfloor - 2 \sum_{1\leq j\leq \frac{\log 2n}{\log p}} \left\lfloor \frac{n}{p^j} \right\rfloor\\ &= \sum_{1\leq j\leq \frac{\log 2n}{\log p}} \left(\left\lfloor \frac{2n}{p^j} \right\rfloor - 2\left\lfloor \frac{n}{p^j} \right\rfloor \right)\end{aligned}$$

Now you'll show on your midterm that $\left\lfloor \frac{2n}{p^j} \right\rfloor - 2\left\lfloor \frac{n}{p^j} \right\rfloor$ is always 0 or 1, so in particular

$$\left\lfloor \frac{2n}{p^j} \right\rfloor - 2\left\lfloor \frac{n}{p^j} \right\rfloor \leq 1 \qquad \forall j.$$

It follows that

$$\nu_p(\binom{2n}{n}) \leq \sum_{1\leq j\leq \frac{\log 2n}{\log p}} \left(\left\lfloor \frac{2n}{p^j} \right\rfloor - 2\left\lfloor \frac{n}{p^j} \right\rfloor \right) \leq \sum_{1\leq j\leq \frac{\log 2n}{\log p}} 1 \leq \frac{\log 2n}{\log p}.$$

The lemma is proved! □