Previously on "Intro to Number Theory"... we proved the following:

Theorem (Chebyshev): $\dfrac{x}{\log x} \ll \pi(x) \ll \dfrac{x}{\log x}$.

By contrast:

Prime Number Theorem: $\pi(x) \sim \dfrac{x}{\log x}$

Recall the difference between the two symbols $\ll$ and $\sim$:

$$f(x) \ll g(x) \qquad \longleftrightarrow \qquad \left| f(x)/g(x) \right| \leq C$$
$$f(x) \sim g(x) \qquad \longleftrightarrow \qquad f(x)/g(x) \longrightarrow 1.$$

Thus, Chebyshev's theorem says that the quantity $\frac{\pi(x)}{x/\log x}$ is bounded between two constants as x increases, whereas the Prime Number Theorem asserts this quantity gets closer and closer to 1; the latter is clearly a much stronger assertion.

The Prime Number Theorem can be expressed in a more quantitative way. To see this, first we write

$$\pi(x) = \frac{x}{\log x} + \mathrm{Err(x)}.$$

The PNT asserts that $\frac{x}{\log x}$ is the main term and that $\mathrm{Err}(x)$ is small relative to the main term, i.e.

$$\lim_{x \to \infty} \frac{\mathrm{Err(x)}}{x/\log x} = 0.$$

It turns out one can prove $\mathrm{Err}(x) \ll x/\log^2 x$, so sure enough, the error in our approximation is small relative to the main term of $x/\log x$. However, it turns out there's a much better approximation to $\pi(x)$:

Prime Number Theorem, v2.0: $\pi(x) \sim \displaystyle\int_2^x \frac{dt}{\log t}$.

Why is this better than the original PNT? Because it has a smaller error term. More precisely, it turns out we have

$$\pi(x) = \int_2^x \frac{dt}{\log t} + \mathrm{Err}(x)$$

with $\mathrm{Err}(x) \ll \frac{x}{(\log x)^{1000}}$. (Here 1000 is chosen because it's Ben's favorite large number – we could replace it with any constant we like.) Thus, the main term in the second version of the PNT is closer to the true value of $\pi(x)$ than the main term in the first version. We'll return to this point below.

But first: why do we write the main term in the second version of the PNT as an integral? i.e. why don't we just integrate and then write the result? It turns out that we can't: amazingly, it's possible to prove that there's no way to express the function $\int \frac{dt}{\log t}$ in terms of any other elementary functions, without resorting to using integral signs or infinite sums. (This remarkable result comes out of a field called the *Galois theory of linear differential equations*. It turns out there are tons of other integrals that arise in nature that we can't evaluate either, e.g. the Gaussian integral $\int e^{-x^2}\,dx$.)

Above we wrote two versions of the PNT. How can they both be true? This can only happen if the following holds:

<u>Proposition:</u> $\displaystyle\int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$.

There are a couple different ways to approach this. One method, suggested by Alex, is to use *L'Hôpital's rule*, which asserts

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

whenever $f(x), g(x) \to 0$ or $f(x), g(x) \to \infty$ as $x \to \infty$.

<u>Proof of Proposition:</u> By L'Hôpital,

$$\lim_{x\to\infty} \frac{\int_2^x \frac{dt}{\log t}}{x/\log x} = \lim_{x\to\infty} \frac{1/\log x}{(\log x - 1)/\log^2 x} = \lim_{x\to\infty} \frac{\log x}{\log x - 1} = 1.$$

Actually, to use L'Hôpital in the first equality above we need to know that the numerator tends to ∞ with x, since we already know (how?) that the denominator does. Akhil pointed out that

$$\int_2^x \frac{dt}{\log t} \geq \int_2^x \frac{dt}{t}$$

and we know that the second integral tends to ∞. $\square$

To see a different approach to proving the proposition, suppose we didn't know that the crazy integral $\displaystyle\int_2^x \frac{dt}{\log t}$ is impossible to integrate. What might we do to try to evaluate the integral? A natural guess is to integrate by parts. Setting $u := 1/\log t$ and $dv := dt$, we find

$$\int_2^x \frac{dt}{\log t} = \frac{t}{\log t}\bigg|_2^x + \int_2^x \frac{dt}{\log^2 t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{\log^2 t}.$$

If we integrate by parts again we get

$$= \frac{x}{\log x} + C_1 + \frac{x}{\log^2 x} + C_2 + 2\int_2^x \frac{dt}{\log^3 t}.$$

In particular, we deduce that

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + \frac{x}{\log^2 x} + \text{smaller terms}.$$

This gives us a better understanding of the error term in the first (crude) version of the PNT. Indeed, putting everything together from above we see that

$$\frac{x}{(\log x)^{1000}} \gg \pi(x) - \int_2^x \frac{dt}{\log t} = \pi(x) - \left(\frac{x}{\log x} + \frac{x}{\log^2 x} + \text{smaller}\right)$$

from which it follows that

$$\frac{x}{\log^2 x} \ll \pi(x) - \frac{x}{\log x} \ll \frac{x}{\log^2 x}.$$

In other words, the fact that the error term in the first (crude) version of the PNT is bounded by $x/\log^2 x$ is really just a statement about the size of the integral $\int_2^x \frac{dt}{\log t}$; it doesn't have anything to do with the nature of primes!

From the second (stronger) version of the PNT, we know

$$\pi(x) - \int_2^x \frac{dt}{\log t} \ll \frac{x}{(\log x)^{1000}}.$$

From computations, however, it's clear that the difference is actually much, much smaller than this. The current conjecture on the size of the error is one of the most famous unsolved problems in all of mathematics:

<u>Conjecture ('Riemann Hypothesis'):</u> $\pi(x) - \int_2^x \frac{dt}{\log t} \ll \sqrt{x}\log x.$

Despite the best efforts of many smart people since Riemann first proposed his conjecture in 1859, we're still very far from being able to prove anything like it. Even a much weaker result, like

$$\pi(x) - \int_2^x \frac{dt}{\log t} \ll x^{0.9999999},$$

would constitute a major breakthrough and would almost certainly win you a Fields Medal.

This ends our excursion into studying primes using calculus. Our next major topic will be a lot less technical:

Modular Arithmetic

Q: What day of the week will it be 8 number of days from today? Tuesday! How about in 28 days? Monday! In 81 days? Friday! How do we know? Well in 77 days, it will be 11 weeks, so it's a Monday again. Then add 4 days, get to Friday!

To formalize this process, it's helpful to set up some numbering system:

Day	Mnemonic	Number
Sunday	Noneday	0
Monday	Oneday	1
Tuesday	Twosday	2
Wednesday		3
Thursday	Foursday	4
Friday	Fiveday	5
Saturday		6

Let's rewrite our last example in terms of this new nomenclature: today is Monday, which is 1. We wish to find the day of the week 81 days from now, which will be $1 + 81 = 82$. Since $82 = 77 + 5$, we see that the answer must be 5, i.e. Fiveday aka Friday.

To demonstrate the power of the numerical approach, we now describe a method for computing the day of the week for any given date. The approach we describe is called the *Doomsday algorithm*; it was invented by John H. Conway. Here are the steps (explained subsequently):

Step 1: Compute the how far the date is from the nearest Doomsday
Step 2: Compute the century day
Step 3: Compute the year÷12 ...
Step 4: then find the remainder...
Step 5: then compute the remainder ÷4.
Step 6: Add up all the numbers to win!

Before describing the steps in detail, we state the main observation underlying the entire algorithm: that in any given calendar year (say, in 2017) the following dates all land on the same day of the week:

$$4/4,\ 6/6,\ 8/8,\ 10/10,\ 12/12 \qquad 9/5,\ 5/9,\ 7/11,\ 11/7 \qquad 3/0$$

Thus, for example, if April 4th happens to fall on a Wednesday some year, then all the other dates in the list above will also fall on a Wednesday that year. (The date 3/0 means the last day of February, which is usually February 28th but during a leap year is February 29th.)

The best way to describe the algorithm is to work a couple examples!

Date: April 9th, 1997 (Miranda's Birthday)
Step 1: The closest doomsday is 4/4. So this date is a Doomsday+5. $\boxed{5}$

Step 2: The century day. This refers to the day of the week the doomsday falls on during the first year of the century. It turns out this obeys a nice rule: the century day falls on the day 7, 5, 3, 2, 7, 5, 3, 2, ... More precisely:

Century	1700	1800	1900	2000	2100	⋯
Century Day	7	5	3	2	7	⋯

(One way to remember this sequence: it's the first four primes in reverse order!) Since Miranda's birthday falls in 19xx, the century day is a $\boxed{3}$.

Step 3: Having taken the century into account, the year is 97. So we have $97 \div 12 = \boxed{8}$...

Step 4: with a remainder of $\boxed{1}$...

Step 5: which we then divide by 4: $1 \div 4 = \boxed{0}$. (Always take the floor.)

Step 6: Finally, add them up: $5 + 3 + 8 + 1 + 0 = 17 = 14 + 3 = 3$. Miranda was born on a Wednesday!

Time for a second example:

Date: February 13th, 2000. (Kimberly's birthday)

Step 1: 3/0 is the closest Doomsday. Since 2000 is a leap year, 3/0 = 2/29. Thus, 2/15 also falls on the Doomsday. Since Kimberly's birthday is 2 days before, our doomsday number is $\boxed{-2}$.

Step 2: The century day is $\boxed{2}$.

Step 3: The year is 00, and $00 \div 12 = \boxed{0}$...

Step 4: with remainder $\boxed{0}$...

Step 5: which divided by 4 yields $\boxed{0}$.

Step 6: Adding up all our numbers gives $-2+2+0+0+0 = 0$. Kimberly was born on a Sunday!

Note that in step 1, we could have subtracted another week to find that 2/8 fell on a Doomsday. This would change our -2 to a 5, but this would not have affected our final answer! This is at the core of modular arithmetic. We'll take up the subject more carefully next time.