Instructor: Leo Goldmakher

NAME: ______________________________

Williams College
Department of Mathematics and Statistics

MATH 313 : NUMBER THEORY

Midterm Exam 1 – due Thursday, October 4th

INSTRUCTIONS:
This exam must be turned in to me in person at the beginning of Thursday's class.
Exams submitted later than this will receive a grade of zero. No exceptions.

ALLOWED: Anything on the course website (including the lecture summaries), the course textbook, your own personal class notes, your own graded homework.

NOT ALLOWED: The internet (apart from the course website or email), books other than the official textbook, interaction with any person other than the instructor.

Please print and sign this page prior to starting your exam. Submit your exam with this cover page affixed.

PROBLEM	GRADE
M.1	
M.2	
M.3	
M.4	
M.5	
M.6	
Total	

I understand that the only aids I may use on this exam are the ones listed above. In particular, I may not use the internet to assist me in any way on this exam apart from accessing the course website or interacting with the instructor. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________

Midterm 1

Please feel very free to reach out to me with questions. Best of luck, and have fun with the problems!

M.1 Prove that $\gcd(n^k - 1, n^\ell - 1) = n^{\gcd(k,\ell)} - 1$ for any positive integers n, k, ℓ.

M.2 Let p_n denote the n-th prime, listed in increasing order (e.g. $p_1 = 2$, $p_2 = 3$, etc). Prove that

$$n \log n \ll p_n \ll n \log n.$$

M.3 Given positive integers a, b, and c, such that $\gcd(a, b) = 1$.

(a) Prove that there exist integers x, y such that $ax + by = c$.

(b) Prove that if $c > ab$, then there exist *positive* integers x, y such that $ax + by = c$.

M.4 Prove that $\binom{2n}{n-3} < \binom{2n}{n}$ for any integer $n \geq 4$.

M.5 Prove that for any $x \geq 1$ we have

$$\lfloor 2x \rfloor - 2\lfloor x \rfloor = 1 \text{ or } 0.$$

M.6 The goal of this question is to obtain an explicit upper bound in Chebyshev's theorem.

(a) By carefully working through our proof of the upper bound in Chebyshev's theorem, prove that $\pi(2^k) \leq 3 \cdot \frac{2^k}{k}$ for all $k \geq 1$. [*Hint: Induction.*]

(b) Deduce from part (a) that $\pi(x) \leq (6 \log 2) \dfrac{x}{\log x}$ for all $x \geq 2$.