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NAME: ______________________________

Williams College
Department of Mathematics and Statistics

MATH 313 : NUMBER THEORY

Problem Set 9 – due Thursday, December 6th

INSTRUCTIONS:
This assignment should be turned in to me in person (i.e. don't leave them in my mailbox or ask someone else to submit on your behalf) at the beginning of Thursday's class. Late assignments may be left in my mailbox (just inside the entrance to Bascom) by **4pm** on Friday; however, 5% will be deducted for late submission. *Assignments submitted later than 4pm on Friday will not be graded.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**GRADE**
9.1	
9.2	
9.3	
Total	

Please read the following statement and sign **before starting this problem set:**

I understand that I am not allowed to use the internet to assist with this assignment. I also understand that I must write down the final version of my assignment in isolation from any other person, and to not copy from any set of written notes created when another person was present. I pledge to abide by the Williams honor code.

SIGNATURE:______________________________

Problem Set 9

9.1 The goal of this exercise is to explore the *partition numbers*, denoted $p(n)$. For each $n \geq 1$, let $p(n)$ be the number of distinct ways of writing n as the sum of a non-increasing sequence of positive integers. For example,

$$5 = \underbrace{5}_{i} = \underbrace{4+1}_{ii} = \underbrace{3+2}_{iii} = \underbrace{3+1+1}_{iv} = \underbrace{2+2+1}_{v} = \underbrace{2+1+1+1}_{vi} = \underbrace{1+1+1+1+1}_{vii}$$

are all the ways to write 5 as a sum of a non-increasing sequence of positive integers, so $p(5) = 7$. We also set

$$p(0) := 1 \qquad \text{and} \qquad p(n) := 0 \quad \forall n < 0.$$

Although the partition numbers may seem random and artificial, they appear in many areas outside number theory, most notably in representation theory, the theory of modular forms, and algebraic combinatorics. Moreover, their study led to some fundamental advances in mathematics; for example, Hardy and Ramanujan invented the 'circle method' (now a major tool in analytic number theory) to study the asymptotic growth rate of $p(n)$.

In this exercise we'll prove the following remarkable recurrence, discovered by Euler:

$$(\heartsuit) \qquad p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) + \cdots$$

Hopefully this looks familiar: recall from class that

$$(\clubsuit) \qquad \prod_{k \geq 1}(1 - x^k) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + \cdots$$

(a) Compute $p(7)$ in two different ways: first by using Euler's recurrence ($\heartsuit$), and then directly from the definition.

(b) Let $b_k := \frac{k(3k-1)}{2}$ denote the pentagonal numbers, and set $b'_k := \frac{k(3k+1)}{2}$. Prove that $b'_k = b_{-k}$ for every $k \geq 1$. Thus, we can write ($\clubsuit$) in the form

$$\prod_{k \geq 1}(1 - x^k) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell x^{b_\ell}.$$

(c) Prove the formula for the sum of a geometric series: whenever $|r| < 1$,

$$\sum_{m \geq 0} r^m = \frac{1}{1-r}.$$

(d) Carefully explain why

$$\prod_{k \geq 1}(1 - x^k)^{-1} = \sum_{j \geq 0} p(j) x^j.$$

[*Hint: use part (c)!*]

(e) Write

$$\left(\sum_{\ell \in \mathbb{Z}} (-1)^\ell x^{b_\ell} \right) \left(\sum_{j \geq 0} p(j) x^j \right) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Show that $c_n = \sum_{k \in \mathbb{Z}} (-1)^k p(n - b_k)$.

(f) Prove Euler's recurrence relation ($\heartsuit$).

9.2 Recall from class that the set $\{(b^2 - a^2, 2ab, b^2 + a^2) : a, b \in \mathbb{Z}\}$ contains all primitive pythagorean triples.

(a) Find an example of pythagorean triple not contained in this set.

(b) Prove that $(b^2 - a^2, 2ab, b^2 + a^2)$ is a primitive pythagorean triple iff $(a, b) = 1$ and a and b have opposite parity.

(c) Prove that every $n \geq 3$ that isn't $\equiv 2 \pmod 4$ is in a *primitive* (nontrivial) pythagorean triple.

(d) Prove that every $n \geq 3$ is in a nontrivial pythagorean triple.

9.3 Find nine distinct rational points on the elliptic curve $y^2 = x^3 + 8$. [*Hint: There are five rational points that are easy to find. Now use these to generate more.*]