A SHORT PROOF OF THE CANTOR-SCHRÖDER-BERNSTEIN THEOREM

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ABSTRACT. We give a relatively short proof of the Cantor-Schröder-Bernstein.

1. STATEMENT AND PROOF

Motivated by Cantor's theory of infinite sets, we write $A \approx B$ to denote the existence of a bijection $A \to B$. In practice it can be quite difficult to construct a bijection between two sets. The Cantor-Schröder-Bernstein theorem[1] is a tool for proving the existence of a bijection without ever having to construct one.

Notation. The symbol $A \hookrightarrow B$ means there exists an injection of A into B, and $A \twoheadrightarrow B$ means there exists a surjection of A onto B. The symbol $X \sqcup Y$ denotes the disjoint union of X and Y, i.e. $X \sqcup Y = X \cup Y$ but also connotes that $X \cap Y = \emptyset$.

Theorem 1 (Cantor-Schröder-Bernstein)**.** *If $A \hookrightarrow B$ and $B \hookrightarrow A$ then $A \approx B$.*

This statement may seem intuitive, but it's surprisingly difficult to prove. I strongly urge the reader to stop reading here and take at least five minutes to take a stab at proving it; this is the best way I know of to appreciate the proof given below.

The heart of the proof is contained in the following special case of the theorem:

Theorem 2. *If $A \hookrightarrow B$ for some $B \subseteq A$, then $A \approx B$.*

Proof. Let's call our injection $f : A \hookrightarrow B$. Our goal is to partition B into two disjoint pieces, say $B := B_f \sqcup \overline{B}$, in such a way that

$$f(\overline{B}) \subseteq \overline{B} \qquad \text{and} \qquad f(A \setminus \overline{B}) = B_f. \tag{1}$$

I first claim that the existence of such a partition of B implies the theorem.

To see this, note that the second condition in (1) tells us that f maps $A \setminus \overline{B}$ surjectively onto B_f. Since we're given that f is injective, we deduce that f produces a bijection between $A \setminus \overline{B}$ and B_f. If f happens to also be a bijection from $\overline{B}$ to $\overline{B}$, we'd be done, since f would then be a bijection of A onto B! However, our first condition in (1) doesn't imply that f is a bijection from $\overline{B}$ to $\overline{B}$. There is one function that is an obvious bijection of $\overline{B}$ onto $\overline{B}$: the identity map on $\overline{B}$. This inspires us to cobble together a function $g : A \to B$ by setting

$$g(x) := \begin{cases} f(x) & \text{if } x \in A \setminus \overline{B} \\ x & \text{if } x \in \overline{B}. \end{cases}$$

I claim that g is a bijection from A onto B.

Note that $g(x) \in B_f$ iff $x \in A \setminus \overline{B}$ and $g(x) \in \overline{B}$ iff $x \in \overline{B}$. From the definition it's clear that g surjects onto B, since it surjects onto each of the two pieces B_f and $\overline{B}$. In particular, $g^{-1}(y)$ is nonempty for all $y \in B$. Now pick any $y \in B$; since $B = B_f \sqcup \overline{B}$, we have $y \in B_f$ xor $y \in \overline{B}$. If $y \in B_f$, then $g^{-1}(y) \in A \setminus \overline{B}$. If $y \in \overline{B}$, then $g^{-1}(y) = y \in \overline{B}$. Since g is injective into each of B_f and $\overline{B}$ individually, we conclude that $g : A \hookrightarrow B$. In other words, g is a bijection from A onto B, so $A \approx B$ as claimed!

[1] So named because it was first proved by Dedekind; see the Wikipedia article for a history of the theorem.

All that remains to do is to define B_f and $\overline{B}$ so that they partition the set B and satisfy the hypotheses (1). We start with the former:

$$B_f := \bigsqcup_{n\geq 1} f^n(A \setminus B),$$

where f^n means $\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$. Then we have

$$f(A \setminus \overline{B}) = f\big((A \setminus B) \sqcup B_f\big) = f(A \setminus B) \sqcup f\left(\bigsqcup_{n\geq 1} f^n(A \setminus B)\right) = B_f.$$

The definition of $\overline{B}$ is now forced upon us:

$$\overline{B} := B \setminus B_f.$$

We need to check that this satisfies (1). Pick any $y \in B_f$. Then $y \in f^n(A \setminus B)$ for some $n \geq 1$, whence

$$f^{-1}(y) \in B_f \sqcup (A \setminus B).$$

Thus for any $x \notin B_f \sqcup (A \setminus B)$ we have $f(x) \notin B_f$. It follows that

$$f(B \setminus B_f) \subseteq B \setminus B_f$$

as claimed. This concludes the proof of Theorem 2. □

Proof of Cantor-Schröder-Bernstein. Given $f : A \hookrightarrow B$ and $g : B \hookrightarrow A$, set

$$A' := g(B) \subseteq A.$$

By Theorem 2, $A \approx A'$. But also, since g is injective, $A' \approx B$. Thus $A \approx B$. □

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