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MATH 350 : REAL ANALYSIS

A Guided Exploration of Compactness

Due Monday, December 9th, at 4pm EST

1 Overview of assignment

The purpose of this assignment is for you to wrestle with, understand, and explain the notion of *compactness*, a concept originating in real analysis that has ended up playing an important role in many other disciplines. Your final draft should be written in IATEX, aimed at an audience of Williams undergraduates who understand the material we've covered in our textbook but don't know any real analysis beyond that.

This is not a free-form essay; rather, it's an exercise in digesting and synthesizing material. Below, I'll describe what resources you may and may not use, and write a bit about the mathematical context. Your assignment begins in the final section, labelled Prompts.

2 Allowed resources

Here's a list of allowed (and not allowed) resources:

- *Internet:* No internet resources, apart from the lecture and problem set material (including the solution sets) posted on the course website and on Glow.
- *Books:* No books apart from the official course textbook (Johnsonbaugh-Pfaffenberger). Moreover, you may only refer to material from chapters 1–19, 22–26, 28, and 30–34 of the text. You may use any results from these chapters without re-proving them.
- *Notes:* You may consult any notes you have taken *in this course*, as well as your submitted problem sets. You may use any results from lectures, problem sets, or the textbook without re-proving them.
- *Humans:* You may (and should!) ask me any questions you have, even if you think they might be overstepping; let me be the judge of which questions I can or cannot answer. That said, you may *not* discuss the problems with any other person until January 2025, orally or otherwise.
- AI: You may not use ChatGPT or any other form of AI on this assignment.

3 Background

Recall that an open interval is any set of the form

$$(a,b) := \{ x \in \mathbb{R} : a < x < b \}$$

where $a, b \in \mathbb{R}$. Similarly, a *closed interval* is any set of the form

$$[a,b] := \{x \in \mathbb{R} : a \le x \le b\}$$

for some $a, b \in \mathbb{R}$. Note that \emptyset is both an open interval and a closed interval. (Make sure you understand why this is!)

The interplay between open and closed intervals is less intuitive than one might initially think. For example:

Exercise 1. Show by example that it's possible for an intersection of open intervals to equal a non-empty closed interval, and for a union of closed intervals to equal a non-empty open interval.

A remarkable theorem (called the *Heine-Borel theorem*) connects open and closed sets in a surprising way. We warm up by stating a special case of it:

<u>THEOREM 1</u>. Suppose $[0,1] \subseteq \bigcup_{\alpha \in A} \mathcal{O}_{\alpha}$, where $A \subseteq \mathbb{R}$ and \mathcal{O}_{α} is an open interval for each $\alpha \in A$. Then there

exists an $N \in \mathbb{Z}_{pos}$ and elements $\alpha_1, \alpha_2, \ldots, \alpha_N \in A$ such that $[0, 1] \subseteq \bigcup_{n=1}^N \mathcal{O}_{\alpha_n}$.

Phrased less formally: if [0, 1] is covered by a bunch of open intervals (possibly uncountably many!), then it's possible to find a finite number of these open intervals that still cover [0, 1].

As alluded to earlier, Theorem 1 is a special case of a more general result. To explain this, we first generalize the notions of open and closed intervals:

<u>DEFINITION</u>. We say a subset of \mathbb{R} is *open* iff it's a union of open intervals. We say a subset of \mathbb{R} is *closed* iff its complement is open.

Remark. Thus, for example, \mathbb{R} is both open and closed.

It might strike you as odd that the definition of closed isn't the direct analogue of the definition of open. There's an excellent reason we don't define closed this way:

Exercise 2. Prove that *any* subset of \mathbb{R} is a union of closed intervals.

Note that some sets are closed, some sets are open, some sets are both, and some sets are neither. Our next goal is to introduce a different type of set—called a *compact* set—that turns out to be very important in practice. The notion of compactness is a bit subtle, so we introduce it in stages.

<u>DEFINITION</u>. Given a set $S \subseteq \mathbb{R}$, an open cover of S is any collection of open sets whose union contains S. More formally, an open cover of S is any collection $\{\mathcal{O}_{\alpha} : \alpha \in A\}$ such that $S \subseteq \bigcup_{\alpha \in A} \mathcal{O}_{\alpha}$, where $A \subseteq \mathbb{R}$ and for

each $\alpha \in A$, \mathcal{O}_{α} is an open set.

In this language, Theorem 1 asserts that for any open cover of [0, 1], there exists a finite subset of the open cover that's also an open cover of [0, 1]. This is the essence of compactness. Here's the general definition:

<u>DEFINITION</u>. A set $S \subseteq \mathbb{R}$ is *compact* iff every open cover of S contains a finite subset that's also an open cover of S. More formally, S is compact iff whenever $S \subseteq \bigcup_{\alpha \in A} \mathcal{O}_{\alpha}$ with $\{\mathcal{O}_{\alpha} : \alpha \in A\}$ a collection of open sets,

$$\exists N \in \mathbb{Z}_{\text{pos}} \text{ and elements } \alpha_1, \alpha_2, \dots, \alpha_N \in A \text{ such that } S \subseteq \bigcup_{n=1}^N \mathcal{O}_{\alpha_n}.$$

Thus, Theorem 1 asserts that [0,1] is compact. We're now in a position to state the generalization of this: HEINE-BOREL THEOREM. A subset of \mathbb{R} is compact iff it is closed and bounded.

4 Prompts

Your assignment should consist of responses to the following questions, in the order they appear below. In addition to being mathematically correct, you should strive for your writing to be as clear and simple as possible. The goal isn't merely to prove that you understand the concepts, but also to explain them well in writing.

Prompt 1. Complete Exercises 1 and 2 from the Background section above.

Prompt 2. Show by example that Theorem 1 would be false if we replaced [0,1] by the open interval (0,1). [Needless to say, you may not invoke the Heine-Borel theorem in your response!]

Prompt 3. Write down a carefully explained proof that the closed interval [0, 1] is compact (i.e. prove Theorem 1). I encourage you to study the proof of Theorem 34.2 given on pages 113-114 of the book, but please use the notation of Theorem 1 and *use your own words* in your proof—if you wanted to explain the proof to someone, how would you do it? Among other things, I strongly urge you to completely reorganize the structure of the book's proof and make it your own.

Prompt 4. Prove that if $S \subseteq \mathbb{R}$ is compact, then S must be bounded. [*Hint: Start by constructing an open cover of S that doesn't depend on S.*]

Prompt 5. The goal of this prompt is to prove that compact sets must be closed. Throughout, assume $S \subseteq \mathbb{R}$ is compact.

- (a) Prove that for any $p \notin S$ and any $x \in S$, there exists $\epsilon > 0$ such that the two open intervals $(p \epsilon, p + \epsilon)$ and $(x \epsilon, x + \epsilon)$ are disjoint (i.e. have empty intersection). [ϵ is allowed to depend on both x and p.]
- (b) Prove that for any $p \notin S$, there exists $\delta > 0$ such that the open interval $(p \delta, p + \delta)$ is disjoint from S. In other words, prove the existence of $\delta > 0$ such that $(p - \delta, p + \delta) \subseteq S^c$. [Use compactness!]
- (c) Deduce from above that S must be closed.