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MATH 350 : REAL ANALYSIS

Final Exam

INSTRUCTIONS

This exam will consist of three questions, to be discussed orally with the instructor. The duration of the exam is expected to be less than 30 minutes. You will have access to a blackboard, as well as the list of questions on the next page; *no other aids are permitted.*

The exam will begin with Question A, which will be asked of every student. The other two questions will be selected by coin flip from Lists B and C, one question from each. (See next page for questions.)

I would like you to understand each topic as deeply as possible. To this end, I reserve the right to follow up on anything you mention during your discussion. For example, if you use the phrase 'infimum of S' I may ask you to define what that is; I may then follow up by asking you to prove that it exists. In short, as you study the material, I want you to continually ask yourself the question: *can I define / prove this without looking it up*?

While I hope you will understand all the details, please avoid explaining all of them unless I ask you to do so. Start any discussion of a proof by explaining a birds'-eye overview of the main steps; if I want more details, I'll ask for them.

Often, it is during an exam that you realize for the first time that you don't properly understand something. This is not only natural, it is totally OK; I will give you as many hints as you need to get back on track. Indeed, the most valuable aspect of an oral exam is that it's a chance for some individualized learning.

I strongly encourage you to practice for the exam with one another; find a study buddy or two and take turns acting the role of the examiner. Don't go easy on your partner! Any time anything is unclear, follow up on it; any time they use a theorem or a definition, ask them for more details. If you don't know other folks in the class or are uncomfortable reaching out to someone, please let me know and I'd be happy to put you in touch with someone else in a similar position.

Best of luck!

-Leo

Problems on other page...

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PROBLEMS

Question A (Big Theorem)
State and prove the Cauchy criterion.

List B (Theory)

B.0 Prove (from the axioms of $\mathbb{R}$) that $1 > 0$.

B.1 Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

B.2 Prove that $\mathcal{P}(X)$ is strictly larger than X for any set X. (Here $\mathcal{P}(X)$ denotes the power set of X.)

B.3 State and prove Bolzano's theorem.

B.4 State and prove the Monotone Convergence Theorem.

B.5 Consider the following:

Claim. *Given $X \subseteq \mathbb{R}$, $(c_n) \subseteq X$ a Cauchy sequence, and $f : X \to \mathbb{R}$ a continuous function on X. Then the sequence $\big(f(c_n)\big)$ is Cauchy.*

"Proof". Given $\epsilon > 0$. Pick any $a \in X$. Because f is continuous at a, there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \epsilon.$$

Since (c_n) is Cauchy, $|c_m - c_n| < \delta$ for all large m, n. Thus $|f(c_n) - f(c_m)| < \epsilon$ for all m, n large. $\square$

Find a counterexample to the claim, and carefully identify the mistake in the alleged proof.

B.6 Given a metric space (X, d), a positive real number r, and a point $\alpha \in X$, prove that the open ball $\mathcal{B}_r(\alpha)$ is an open set.

B.7 Suppose $(a_n), (b_n)$ are convergent sequences, and write $a_n \to A$ and $b_n \to B$. Prove that $a_n b_n \to AB$.

List C (Computation)

C.0 I will give you a sequence and ask you to determine whether or not it converges (with proof).

C.1 I will give you one of three series ($\sum\limits_{n=1}^{\infty} \frac{1}{n}$, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, or $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$) and ask you to determine whether or not it converges (with proof).

C.2 I will give you a function f and a number a and ask you to determine whether $\lim\limits_{x \to a} f(x)$ exists or not (with proof).

C.3 Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{a}{n} \text{ in reduced form.} \end{cases}$$

Where is f continuous, and where is f discontinuous? Prove it.