September 5, 2024

MATH 350: Lecture 1

Please refer to the professor of this course as "Les" or "Professor/Doctor Goldmakher," never simply "Professor."

Notation:

$\mathbb{R}$ - the set of real numbers
$\mathbb{Z}$ - the set of integers (from the German "Zahl")
$\mathbb{Q}$ - the set of rational numbers ($\mathbb{Q}$ for "quotient")

What's a set?

Jackson offered "a collection of elements," and Noam defined a collection as "a set."
→ Seems a little circular...

Answer: we'll come back to that... there's a whole field of set theory which deals with that question.

Instead, we'll do "naïve set theory," where we just take a "set" to be a set as we think of it intuitively.

Notes:

- A set doesn't see repeats
  ↳ e.g. $\{1, 2, 2, 3\} = \{1, 2, 3\}$
- A set doesn't see order
  ↳ e.g. $\{1, 2, 3\} = \{2, 1, 3\}$.

What is a real number?

- Juan says "anything on the real number line"
- Lily says "anything with a positive square."
  - ↳ Pedro suggests saying "nonnegative" to include zero.
  - ↳ Evan asks what about $(3+2i)^2 = 5+12i$.!
    - ↳ begs the question of what it means to be nonnegative.

Let's try something else...

Leo asks Keel, "What's an elephant?" to which he replies, "a large animal with an elongated nose, grayish thick skin." Thomasina says that we may need some more descriptors to completely differentiate elephants from all other animals.

But it does seem we can define something in terms of its properties rather than "what it is."

For a mathematical example, Leo asks, "What is $\sqrt{2}$?"

- Alice says it's the number that when you multiply it by itself, you get 2.

But others point out $-\sqrt{2}$ satisfies this property, so we may want to add "positive."

- Leo also adds that this doesn't give us any information about the number (e.g. its size).

But we will use this way of defining something by listing properties to define $\mathbb{R}$. (it turns out 13 properties suffice!)

Wyatt notes that elephants are real things we can observe; while numbers are not and may therefore follow different rules.

In many ways math and science are opposites of each other. Science tries to understand the laws of the universe, while math starts with the rules.

Back to $\mathbb{R}$, in this course we will prove that given our list of 13 properties completely determines $\mathbb{R}$ (i.e. any set that satisfies them is morally the same as $\mathbb{R}$), and that removing any of them will result in other possible sets (i.e. list isn't redundant).

"Bitte vergiß alles, was Du auf der Schule gelernt hast, denn Du hast es nicht gelernt" - Edmund Landau

Translation: "Please forget everything that you have learned in school, because you have not learned it."

↳ In this class, we come with our own intuitions for and ideas about the real numbers, but here we will look at them in a totally new way.

Back to naïve set theory...

Given two sets A and B, we can combine them in various ways to form new sets

Examples: $A = \{1, 2, \text{elephant}\}$

$B = \{2, \text{apple}, \text{Nathan}\}$

"Union of A and B": $A \cup B = \{1, 2, \text{elephant}, \text{apple}, \text{Nathan}\}$

"Intersection of A and B": $A \cap B = \{2\}$

$\{A, B\} = \{\{1, 2, \text{elephant}\}, \{2, \text{apple}, \text{Nathan}\}\}$

Empty set: $\emptyset = \{\}$, the set consisting of nothing

Note: $\{\emptyset\}$, the set containing the empty set, is a set consisting of one object, as opposed to the empty set with no elements.

"Set minus": $A \setminus B :=$ "the set of all elements in A that aren't in B"

$A \setminus B = \{1, \text{elephant}\}$

Notation: $X := Y$ means X is defined to be Y.

"Power set of A": the set of all subsets of A, or in math lingo, $\mathcal{P}(A) := \{S : S \subseteq A\}$

("$\subseteq$" is "is a subset of")

e.g. $\mathcal{P}(\{1,2,3\}) = \{\{1\}, \emptyset, \{1,2,3\}, \{1,3\}, \{3\}, \{2\}, \{1,2\}, \{2,3\}\}$

Notation: $x \in A$ means "x is an element of A"

e.g. $\text{elephant} \in \{1, \text{elephant}\}$

$\emptyset \notin \{1,2\}$, but $\emptyset \subseteq \{1,2\}$.

Already during the first class, we can now understand an open problem in the field of set theory!

Conjecture (Frankl, 1979): Suppose A is a finite set whose elements are all finite sets, that is closed under unions (i.e. if $B, C \in A$, then $B \cup C \in A$), then A has a "popular element," i.e. there exists an element X that lives in $\geq 50\%$ of the elements of A.

Gilmer proved this with 50% replaced by 1%, which was subsequently improved to 38%.