# MATH 350: LECTURE 2

SUMMARY. We reviewed some of the sets we defined last time and introduced a few new ones. We then found a way to define ordered pairs using sets only and began exploring how to formalize our intuition about functions using these tools from set theory.

### 0. Preliminaries

After some announcements, Leo answered some housekeeping questions:

- **Q**: Can we email tex'd up HW?
- A: Print out for now, but will think more about it.
- **Q**: What is an expository essay?
- A: "An essay where you exposit." More details to come later, but you will explain in your own words an analysis topic that we did not study in class.
- **Q**: Precept meetings?
- A: Groups of 3 meet with a TA one hour each week. You take a short quiz where you prove a theorem from class. Each person presents a problem from previous problem set.
- **Q**: Content for oral final?
- A: You'll get a list of possible problems in advance, you prepare answers, then you will get three problems at random in your exam.
- **Q**: Will there be designated problems to tex up on each problem set?
- A: There are no tex'd problems this week, next week there will be one problem, and then the number will be bumped up as time goes on. Only these tex'd problems will be graded for correctness; the other problems will be graded for completeness.
- **Q**: Does the principle of explaining what we did, what we tried, what we need to prove, etc, from Precept meetings apply to psets and midterms?
- A: Yes.
- **Q**: Can we bring notes/psets to precept meetings?
- **A**: The idea is to explain your solutions without notes. Aim to understand rather than memorize. You will not have notes in your oral exam.

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Template by Leo Goldmakher.

### 1. LAST TIME

Recall we've been using naive set theory. We assume we know what a set is and nothing else. Everything will be built from sets. Given sets A, B, we formed

(Union) 
$$A \cup B := \{x : x \in A \text{ or } x \in B\},$$
  
(Intersection)  $A \cap B := \{x : x \in A \text{ and } x \in B\},$   
(Set difference)  $A \setminus B := \{x \in A : x \notin B\}.$ 

Inspired by observations from Leila and Cameron, we realized we could reformulate the definition of intersection:  $A \cap B = \{x \in A : x \in B\}.$ 

We also defined the **power set**  $\mathcal{P}(A) := \{S \subseteq A\}$ , the set of all subsets of A. Note that we use  $\subseteq$  to denote any subset, while  $\subset$  denotes a *proper* subset, i.e. a subset that is strictly smaller.

#### 2. More about sets

We introduced some new sets, starting with the **complement** of a set.

**Definition.** The complement of A is defined to be  $A^c := \{x \notin A\}$ .

**Example.** Let  $A = \{1, 2, 5\}$ . Then  $A^c$  is ...??

Jenny pointed out that it doesn't really make sense to ask what's not in A if we don't know where the elements of A are coming from. To address this, we specify our **universe** U of all allowed elements.

Example. If

$$U := \{1, 2, 3, 4, 5, 6\}$$

then

$$A := \{2, 3, 5\} \implies A^c = \{3, 4, 6\}$$

Thomasina noted that  $A^c = U \setminus A$ . Juan claimed:

Claim.  $A \cup A^c = U$ .

*Proof.*  $(\Rightarrow) A \cup A^c \subseteq U$ . Let  $x \in A \cup A^c$ . Then  $x \in U$ . This direction is basically trivial since we require x to come from U in the first place.  $(\Leftarrow) U \subseteq A \cup A^c$ . Pick  $x \in U$ . If  $x \in A$ , then  $x \in A \cup A^c$ . If  $x \notin A$ , then  $x \in A^c$ , so  $x \in A \cup A^c$ .

This proof illustrates a standard technique for showing equality of sets: prove containment in both directions! Note that it is common practice to mark the end of a proof with  $\Box$  or "QED" (from the Latin *quod erat demonstrandum* = "that which was to be demonstrated").

Finally, we defined the cartesian product

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

But Wyatt pointed out that we haven't rigorously defined what an **ordered pair** (a, b) is yet; we've just given a definition in terms of notions we have intuition about. We say a definition like this is **meta-analytic**.

**Question:** What is an ordered pair? What does (a, b) mean in terms of sets?

**Attempt:** Maybe  $(a, b) := \{a, b\}$ ?

Harris noted that since **sets don't recognize multiplicity**, ordered pairs consisting of a repeated element would reduce to a set with only a single element, e.g.  $(1,1) = \{1,1\} = \{1\}$ . But this is actually OK: given  $\{1\}$ , we still uniquely recover the ordered pair (1,1).

But, as Armie pointed out, sets don't recognize order either. So we get  $(1, 2) = \{1, 2\} = \{2, 1\} = (2, 1)$ , which doesn't work because because  $(1, 2) \neq (2, 1)$ . To encode order in our set, we introduced a trick. Define

$$(a,b) := \{\{a,b\},\{a\}\}.$$

The subset with two elements tells us the elements in our ordered pair, and the subset with one element tells us which comes first. For example,

$$(2,1) := \{\{2,1\},\{2\}\} = \{\{1,2\},\{2\}\}$$
$$(3,-1) := \{\{3\},\{-1,3\}\}.$$

Nathan asked why we have the *set* of one element and not just the element itself, i.e. why can't we define  $(a, b) := \{\{a, b\}, a\}$ ? As we will see later, it is desirable to have sets consisting of the same *type* of object. Utsav wondered if we could alternatively specify the *second* element in the ordered pair. It turns out this would also work, but by convention we specify the first element. William pointed out that since sets don't recognize multiplicity, we get things like

which seems a little weird, but it still unambiguously defines an ordered pair, so it's OK! In fact, we'd canonically use  $\{\{1\}\}$  to denote (1, 1) since it is the simplest form, the same way we'd prefer to use the fraction 1/2 instead of 2/4 even though they are the same thing. Cameron remarked that using the same types of objects in our set is what allows us to simplify the set like this, providing one answer to Nathan's earlier question.

But how would we define ordered tuples with *more* than two elements? We'll come back to this later.

# 3. Functions

With help from Ethan and Lily, we came up with a meta-analytic definition for a function.

**Definition** (Meta-analytic). We say  $f : A \to B$  is a **function** from A to B if for every input  $x \in A$ , there is only one output  $y \in B$ .

We might say a function changes x into y or maps A to B. How do we define this transformation analytically, i.e. in terms of sets? Consider the example:

$$f: \mathbb{Z} \to \mathbb{R}$$
$$x \mapsto x^2$$

Following suggestions from Blayne and Noam, we realized we could graph the function and *define* the function as the points on its graph. For example, from our function above we can plot the points



so  $f = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4) \dots\}$ . We can formalize this as follows.

**Definition** (2.0). We say  $f : A \to B$  is a function if and only if  $f \subseteq A \times B$  such that for each  $a \in A, \exists$  exactly one  $b \in B$  with  $(a, b) \in f$ .

Then, if  $(x, y) \in f$ , we write f(x) = y.

Ethan pointed out that this definition uses the number "one", which may be problematic since we haven't defined numbers yet—but we will come back to this later. Pedro pointed out that here we were able to define our function neatly with a simple equation. But what about, say, functions that we might define differently on different subdomains? Leo noted that our definition using only the points on the graph permits this as it does not require us to give an equational form of the function.