REAL ANALYSIS: LECTURE 2

SEPTEMBER 11TH, 2023

1. Preliminaries

After a few announcements, we dove right back into things. Last time we talked about a lot of things: sets, real numbers, functions, ordered pairs, and elephants?? We then recapped various set operations. For the following, let A, B be sets in a universe U. Recall that the universe is the set of all allowable things, which implies $A, B \subseteq U$. Then we have the following:

(i) Set Union: $A \cup B := \{x \in U \ : \ x \in A \text{ or } x \in B\}$
(ii) Set Intersection: $A \cap B := \{x \in U \ : \ x \in A \text{ and } x \in B\}$
(iii) Set Difference: $A \setminus B := \{x \in A \ : \ x \notin B\}$
(iv) Set Complement: $A^c := \{x \in U \ : \ x \notin A\}$ (i.e. $A^c := U \setminus A$)
(v) Cartesian Product: $A \times B := \{(a, b) \ : \ a \in A, b \in B\}$
(vi) Power Set: $\mathcal{P}(A) := \{S \subseteq U \ : \ S \subseteq A\}$

A quick remark is that A^c is denoted A' in the textbook. We also reviewed our definition of ordered pairs:

(i) Original Definition: $(a, b) := \{a, b, \{a\}\}$
(ii) Matt's Revamped Definition: $(a, b) := \{b, \{a\}\}$
(iii) Book's Definition: $(a, b) := \{\{a, b\}, \{a\}\}$

Jenna pointed out an interesting issue with the Matt's revamped definition: what's the interpretation of $\{\{a\}, \{b\}\}$? There are two possible interpretations of this!

The main advantage of the book's definition of the others is that all objects in the set are of the same type: they're all sets. Thus, **we will henceforth use the book's definition** of ordered pairs. Let's do an example. What is $(1, 1)$ in terms of our new definition? Noah showed it was the following:

$$\begin{aligned} (1,1) &:= \{\{1,1\}, \{1\}\} \\ &= \{\{1\}, \{1\}\} \\ &= \{\{1\}\}. \end{aligned}$$

It may seem that starting all the way from scratch is putting us way way behind even arithmetic, but in fact we have all the tools to start hacking away on new math. Consider the following conjecture (open problem or problem with no solution yet):

Conjecture 1 (Frankl, 1979)**.** *Suppose S consists of finitely many finite sets, and is closed under unions, i.e. for any $A, B \in S, A \cup B \in S$ (so S is a set of sets, and if you "combine" (union) any two sets, that new set is also in S). Then, there exists a popular element; in other words, there exists an element x such that x is an element of at least 50% of elements in S.*

Although this remains open, last year there was a big breakthrough on this problem by Justin Gilmer, who is not a professional mathematician (he works at Google): he proved that the conjecture holds if you replace 50% with 1%. Five days after he posted his work on the arXiv (a freely accessible online repository for math and physics papers), three different sets of authors posted papers improving his result from 1% to 38%. However, the conjectured 50% remains out of reach, for now.

This conjecture actually has nothing to do with class, but it's just an interesting problem. So, let's go back to actual real analysis: can we define functions in a rigorous manner? Yes. Yes we can.

Notes on a lecture by Leo Goldmakher; summary written by Justin Cheigh.

2. FUNCTIONS

Jon, Miles, Sarah, and some others I may have missed all pitched ideas that led to the following definition.

Definition (Function)**.** Given sets A, B, we say f is a *function* from A to B, denoted $f : A \to B$, iff $f \subseteq A \times B$ s.t. $\forall a \in A, \exists! b \in B$ with the property that $(a, b) \in f$.

An equivalent formulation:

Definition (Function)**.** Given sets A, B, we say f is a *function* from A to B, denoted $f : A \to B$, iff $f \subseteq A \times B$ s.t. $\forall a \in A, \exists! x \in f$ with $\{a\} \in x$.

Without symbols this says "f from A to B is a function if and only if f is a subset of $A \times B$ (i.e. think of functions as a graph) such that every element a of A appears as the first coordinate of exactly one point on f."

Here Gabe asked an interesting question: we wrote $f \subseteq A \times B$, but would we even ever have $f = A \times B$? Alex gave an interesting-er meta-analytic counterexample: consider $A = \{1\}, B = \{1\}$.

Let's go over some basic concepts about functions. Let $f : A \to B$ be a function.

(i) If $(a, b) \in f$, we say $b = f(a)$.
(ii) A is called the *domain* (stuff you put in)
(iii) B is called the *codomain* (stuff you get out)
(iv) Image of f: $f(A) := \{f(a) \ : \ a \in A\}$
(v) Inverse Image of S under f: $f^{-1}(S) := \{x \in A \ : \ f(x) \in S\}$

Here's a meta-analytic example. Let $f : \mathbb{Z} \to \mathbb{Z}$ such that $x \mapsto x^2$. Then we have the following:

(i) Domain of f: $\mathbb{Z}$
(ii) Codomain of f: $\mathbb{Z}$
(iii) $f(\mathbb{Z}) = \{0, 1, 4, 9, \ldots\}$ (image is perfect squares)
(iv) $f^{-1}(\{0\} = \{0\}$ (only 0 maps to 0)
(v) $f^{-1}(\{1\}) = \{1, -1\}$
(vi) $f^{-1}(\{-1\}) = \emptyset$

Notice that the codomain and the image need not be equal (though you may notice the image of f is a subset of codomain). It may seem that images are a much nicer object that inverse images, but this is untrue in one particular way: inverse images "play nice" with set operations.

Proposition 1.

$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$$

$$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$$

On the other hand, this is **not** true for the image. Here's a meta-analytic example (with $f(x) = x^2$) provided by Jeremy:

$$f(\{2\} \cap \{-2\} = f(\emptyset) = \emptyset.$$

On the other hand,

$$f(\{-2\}) \cap f(\{2\}) = \{4\} \neq f(\{2\} \cap \{-2\}.$$

So, intersection doesn't work, but does union? Well, yes. Formally,

Proposition 2. $f(X \cup Y) = f(X) \cup f(Y)$.

With that we ended class. Next class mission: formulate a set of axioms that uniquely define $\mathbb{R}$ (the real numbers).

3. Quick Recap

We started class by reviewing set operations: union, intersection, difference, complement, Cartesian product, and power set. After this we revamped our definition of ordered pairs to the following:

$$(a, b) := \{\{a, b\}, \{a\}\}.$$

We then returned to functions and provided the following rigorous definition:

Definition (Function)**.** Given sets A, B, we say f is a *function* from A to B, denoted $f : A \to B$, iff $f \subseteq A \times B$ s.t. $\forall a \in A, \exists! x \in f$ with the property that $\{a\} \in x$.

We defined some basic objects associated to functions, e.g. the domain, codomain, image, and inverse image. The inverse image turns out to play nice with set operations, while the image doesn't always.