MATH 350: LECTURE 6

1. LAST TIME

Recall we added (A12) to our list of axioms characterizing \mathbb{R} :

(A12) $\exists \mathbb{P} \subseteq \mathbb{R}$ (which we call the positive reals) s.t.

- (i) if $x, y \in \mathbb{P}$ then $x + y \in \mathbb{P}$ and $x \cdot y \in \mathbb{P}$ (the positive reals are closed under addition and multiplication)
- (ii) Trichotomy: $\forall x \in \mathbb{R}$ exactly one of the following holds:

$$x = 0 \qquad x \in \mathbb{P} \qquad -x \in \mathbb{P}$$

We then saw that this notion of positivity induces an order on \mathbb{R} .

Definition. x < y iff $y - x \in \mathbb{P}$.

We can use this definition to easily recover many familiar properties, e.g.,

Proposition 1.1. If a > 0 and x > y, then ax > ay.

Proof.
$$a, x - y \in \mathbb{P} \implies a(x - y) \in \mathbb{P} \implies ax - ay \in \mathbb{P} \implies ax - ay > 0.$$

At this point, we've eliminated many of our impostors, including \mathbb{F}_2 and \mathbb{C} . The most familiar sets satisfying (A1) - (A12) are \mathbb{Q} and \mathbb{R} . Some other sets we considered:

- $\mathbb{R} \setminus \mathbb{Q}$ (the irrational numbers)? Cameron pointed out that the set of irrationals isn't closed under multiplication since $\sqrt{2} \cdot \sqrt{2} = 2$. It is also not closed under addition: $\sqrt{2} + (-\sqrt{2}) = 0$.
- (Noam) $\{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. It turns out this does satisfy (A1) (A12)!

2. Distinguishing \mathbb{R} from \mathbb{Q}

Question: What distinguishes \mathbb{R} from \mathbb{Q} ? What does \mathbb{R} have that \mathbb{Q} doesn't?

Theorem 2.1. $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

Proof (Meta-analytic). Suppose $\sqrt{2} \in \mathbb{Q}$, say $\sqrt{2} = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. Then

$$2 = \frac{a^2}{b^2} \implies 2b^2 = a^2 \implies a^2 \text{ is even} \implies \boxed{a \text{ is even}} \implies a = 2c \text{ for some } c \in \mathbb{Z}$$
$$\implies 2b^2 = a^2 = (2c)^2 = 4c^2 \implies b^2 = 2c^2 \implies b^2 \text{ is even} \implies \boxed{b \text{ is even}}$$

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In words, we've proved:

If we express $\sqrt{2}$ as a fraction, both the numerator and denominator must be even. (\clubsuit) By itself, this isn't a contradiction: $\frac{2}{4}$ is a perfectly reasonable fraction with both numerator and denominator being even. However, given any such fraction, we can always reduce it:

$$\sqrt{2} = rac{a/2}{b/2}$$
 with $a/2, b/2 \in \mathbb{Z}$

But now (\clubsuit) applies again, so both $\frac{a}{2}$ and $\frac{b}{2}$ must be even, implying

$$\sqrt{2} = rac{a/4}{b/4}$$
 with $a/4, b/4 \in \mathbb{Z}.$

We can repeat this process as many times as we'd like; in particular, for any positive integer k we deduce

$$\frac{a}{2^k}, \frac{b}{2^k} \in \mathbb{Z}$$

But the only integer divisible by arbitrarily large powers of 2 is zero, so a = b = 0. This contradicts our initial assumption that $b \neq 0$, and we conclude.

Remark. We could start by assuming a/b is an irreducible fraction, which means we arrive at a contradiction when we find that both a and b are even. The advantage of the above approach is that it doesn't require us to make that somewhat mysterious starting assumption.

Here is an alternate proof:

Proof 2 (meta-analytic). Suppose $\sqrt{2} = \frac{a}{b}$. Then we have $2 = \frac{a^2}{b^2} \implies b^2 \mid a^2$ (i.e., a^2 is a multiple of b^2) $\implies b \mid a \implies \sqrt{2} = \frac{a}{b} \in \mathbb{Z}$. But $1 < \sqrt{2} < 2$.

One happy feature of this proof is that, with very little modification, it produces a much more general result:

Proposition 2.2. For any $n \in \mathbb{Z}$, either $\sqrt{n} \in \mathbb{Z}$ or $\sqrt{n} \notin \mathbb{Q}$.

Clearly \mathbb{R} and \mathbb{Q} are different since $\sqrt{2}$ is in \mathbb{Q} but not in \mathbb{R} . But what sort of axiom could we introduce that would account for this distinction? Intuitively, we know the real line should be "continuous", while we've just seen that \mathbb{Q} has "holes" in it, e.g., at $\sqrt{2}$. Schematically,



How might we go about approximating $\sqrt{2}$? William suggested we pick a number, square it, and then repeatedly refine our estimate based on whether the result is less than or greater

than 2. For example,

$$(1.3)^2 = 1.69 < 2$$

$$(1.4)^2 = 1.96 < 2$$

$$(1.5)^2 = 2.25 > 2$$

$$(1.41)^2 < 2$$

$$(1.42)^2 > 2$$

$$\vdots$$

Really what we're doing here is taking a sequence of rational numbers (recall that terminating decimals must be rational!) that gets closer to $\sqrt{2}$ from the left (or the right). This process actually motivates a way to define $\sqrt{2}$.

Idea: Consider $\mathcal{A} := \{x \in \mathbb{Q} : x^2 \leq 2\}$. Then maybe $\sqrt{2} := \max \mathcal{A}$? But this isn't quite right. All of the elements in \mathcal{A} are rational, so the maximum would also be rational, but we've just proved that $\sqrt{2}$ is not rational. Rather than thinking about $\sqrt{2}$ as the largest object in the set, we can think of it as the smallest object bigger than the set. We formalize this idea in the following axiom:

(A13) Given any nonempty $A \subseteq \mathbb{R}$ that is bounded above, $\exists a \in \mathbb{R}$ that's a least upper bound on A.

What does it mean to be a least upper bound?

Definition (Wyatt + Nathan). We say $w \in \mathbb{R}$ is an **upper bound** of S iff $w \ge x \forall x \in S$. We say n is a **least upper bound** of S iff n is an upper bound of S and $n \le w \forall$ upper bounds w.

Example 1. Consider the closed interval $[0,1] := \{x \in \mathbb{R} : 0 \le x \le 1\}$. This set has many upper bounds: $2, 1, \pi^e, \cdots$. What about the least upper bound? William suggested 1, but how do we know this?

Proof (Jenny). 1 is the largest element in [0, 1], hence an upper bound. Therefore, any x < 1 is strictly smaller than something in the set, so it cannot be an upper bound of [0, 1]. \Box

This led Juan to ask about the open interval:

Example 2. What is the least upper bound of $(0,1) := \{x \in \mathbb{R} : 0 < x < 1\}$? Armie proposed that the least upper bound is still 1. Clearly 1 is an upper bound on the set, so we just need to show that anything strictly smaller than 1 can't be upper bound. Harris provided the following proof:

Proof. Pick any x < 1. If $x \le 0$, it isn't an upper bound, so we may assume x > 0. Problem (6) in Problem Set 3 implies the existence of some $\alpha \in \mathbb{R}$ with $x < \alpha < 1$. This implies $\alpha \in (0, 1)$ and is strictly larger than x, which means x cannot be an upper bound of (0, 1). \Box

NOTATION. The least upper bound of \mathcal{A} is denoted lub \mathcal{A} in the book, but this is weird and unappetizing, so we will denote it sup \mathcal{A} (short for "supremum"), which is the conventional term used today.

Remark. The above examples illustrate an important point about the supremum: $\sup \mathcal{A}$ might live in \mathcal{A} or it might not live in \mathcal{A} :

 $\sup(0,1) = 1 \notin (0,1) \qquad \qquad \sup[0,1] = 1 \in [0,1].$

Denis asked how this axiom eliminates \mathbb{Q} . Observe that (A13) tells us that for any subset of \mathbb{R} that has a supremum, i.e. that the supremum lives in \mathbb{R} . From our above example, we have $\{x \in \mathbb{Q} : x^2 \leq 2\} \subset \mathbb{Q}$, but the supremum of this set is $\sqrt{2}$, which does not live in \mathbb{Q} , hence $\mathbb{Q} \neq \mathbb{R}$.