REAL ANALYSIS: LECTURE 7

SEPTEMBER 28TH, 2023

1. Preliminaries

We began with a brief remark about proofs:

Remark. Suppose you wanted to prove that $A \implies B$. There are two general options:

(1) Suppose A. Super smart math stuff... And therefore, by the quantum crypto AI theorem, B. This is a **direct proof**.
(2) Suppose not B. Then equally super smart math... Therefore, not A. This is a **contrapositive proof**.

This is basically saying (if A then B is equivalent to if not B then not A). In fancy math language,

$$(A \implies B) \iff (\neg B \implies \neg A),$$

where $\neg A$ is the *negation of* A, i.e. not A.

Notice that you cannot suppose B and prove A!

This doesn't work out. Here's an example. If you are a math major, then you take real analysis. Notice that the *converse* is: if you take real analysis, you are a math major. This is not true! Plenty of non-math majors take real analysis.

Previously, on real analysis. We discussed induction and proved 1 is the smallest element of $\mathbb{Z}_{\text{pos}}$ and $\mathbb{Z}_{\text{pos}}$ are closed under $+$. Here's a statement similar to induction:

2. Strong Induction

Proposition 1 (Strong Induction)**.** *Suppose $S(n)$ is a sequence of logical assertions, one for each $n \in \mathbb{Z}_{\text{pos}}$, such that*

(i) $S(1)$ is true, and
(ii) $S(k)$ must be true whenever $S(j)$ is true $\forall$ positive integers $j < k$.

Then, $S(n)$ is true $\forall n \in \mathbb{Z}_{\text{pos}}$.

Notice here you have a *stronger* requirement than in ordinary induction: rather than requiring the single condition $S(k-1)$ being true to deduce the truth of $S(k)$, we now need to know that all the statements $S(j)$ with $j < k$ are true in order to imply the truth of $S(k)$.

Let's show an example of using strong induction. We begin with a definition:

Proposition 2. *$\mathbb{Z}_{\text{pos}}$ is well-ordered.*

To make sense of this, we first have to define what it means to be well-ordered:

Definition (Well-Ordered)**.** A set $S \subseteq \mathbb{R}$ is *well-ordered* iff every nonempty subset of S has a least element.

Thanks to Jon for the catch that the subset can't be empty. In other words, a set $S \subseteq \mathbb{R}$ is well-ordered if you can "order" every nonempty subset of S. For example, $\{1, 2\} \subseteq \mathbb{R}$ is well-ordered. Miles gave an example that isn't well-ordered: $\mathbb{R}$ itself. Notice $\mathbb{R} \subseteq \mathbb{R}$, but $\mathbb{R}$ doesn't have a least element.

What about $[0, 1]$? Is it well-ordered? Nope: Harry pointed out that $(.5, .6) \subseteq [0, 1]$ has no least element. Another example of a set that isn't well-ordered (provided by Lexi) is $\mathbb{Z}$; there's no smallest integer.

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

Proof of Proposition 2. We proceed by strong induction, which means we need to come up with a sequence of logical assertions and show the relevant conditions hold.

Suppose $A \subseteq \mathbb{Z}_{\text{pos}}$ has no least element. Let $S(n)$ be the logical assertion $n \notin A$. We proved 1 is the least element in $\mathbb{Z}_{\text{pos}}$. So, if $1 \in A$, 1 would be the least element of A. Thus, $1 \notin A$, i.e. $S(1)$ is true.

Suppose $S(j)$ is true for every positive integer $j < k$. This means that $j \notin A \, \forall j < k$, or equivalently, that every element of A must be at least as large as k. Thus if $k \in A$, it would be the least element of A! This would contradict the definition of A, so $k \notin A$. In other words, we've deduced $S(k)$ is true. By strong induction, $S(n)$ is true for every positive integer n, which means $A = \emptyset$. Thus, $\mathbb{Z}_{\text{pos}}$ are well ordered! □

In general, it makes sense to use strong induction when you must know all of the previous $S(j)$, not just the last piece of information $S(k-1)$.

Let's take a step back. We constructed $\mathbb{R}$, and from there created $\mathbb{Z}_{\text{pos}}, \mathbb{Z}, \mathbb{Q}$. However, we don't really know about relationships between these. Our intuition, on the other hand, understands strong relationships between these sets. For example, we know intuitively that every $x \in \mathbb{R}$ can be approximated pretty well by an integer. Let's prove a formalized version of this:

Proposition 3. *$\forall x \in \mathbb{R}, \exists N \in \mathbb{Z}$ and $\alpha \in [0, 1)$ s.t.*

$$x = N + \alpha.$$

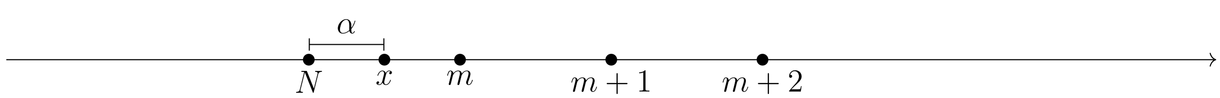
Moreover, N and α are uniquely determined by x.

For example, $\pi = 3 + 0.1415926\dots$ (here $N = 3, \alpha = 0.14159$). Further, the only (N, α) that fulfill the above proposition are 3 and 0.14159; these choices of N and α are *uniquely* determined by x. Intuitively, how can we prove this?

Well, given x, how can we find N and α.

Jenna proposed a nice argument for existence of N and α. Consider all the integers $> x$. By well-ordering there's a least integer m in this set. Then $m - 1$ is the greatest integer $\leq x$, so $N = m - 1$. Then we can find α by $\alpha = x - N$.

Next, Edith suggested a nice argument for uniqueness. Suppose there are two pairs $(N, \alpha), (M, \beta)$ with $x = N + \alpha = M + \beta$. If N and M are different, then their difference is at least 1, but the difference between α and β is at most 1!



Ok, let's prove this.

Proof. We'll assume $x \geq 1$, and you'll prove $x < 1$ on your homework. Let

$$J := \{n \in \mathbb{Z}_{\text{pos}} \;:\; n > x\}.$$

Since $\mathbb{Z}_{\text{pos}}$ are well ordered, J has a least element $m \in \mathbb{Z}_{\text{pos}}$. Then set $N := m - 1, \alpha = x - N$. We claim that

(i) $N \in \mathbb{Z}_{\text{pos}}$
(ii) $N \leq x$
(iii) $\alpha \in [0, 1)$

Proof of (i): We know $m \in \mathbb{Z}_{\text{pos}}$. Here's a quick lemma:

Lemma 1. *If $m \in \mathbb{Z}_{\text{pos}}, m - 1 \in \mathbb{Z}_{\text{pos}} \cup \{0\}$.*

Proof. Proved in Chapter 6 of the book. □

Proof of (i): By the above lemma it suffices to show $N \neq 0$. Since $x \geq 1$, we get $m > x \geq 1$, which means $N = m - 1 > 0 \implies N \neq 0$. So, $N \in \mathbb{Z}_{\text{pos}}$.

Proof of (ii): Notice $m > m - 1 = N \notin J$ but is in $\mathbb{Z}_{\text{pos}}$, which means $N \leq x$.

Proof of (iii) $\alpha = x - N \geq 0$. Also,

$$\begin{aligned} \alpha &= x - N \\ &= x - m + 1. \end{aligned}$$

Since $x < m$, we get $x - m < 0$, which from above tells us $x - m + 1 < 1$.

It looks like we're done with the proof of existence of N and α, but Miles pointed out a fundamental issue. Our entire argument rested on finding the least element of J, but to be able to do this we need to know that $J \neq \emptyset$, which we never proved! In essence, the claim is that there are arbitrarily large positive integers. We'll prove this next class $\square$