REAL ANALYSIS: LECTURE 8

OCTOBER 2ND, 2023

1. Preliminaries

Recall that last time we were approximating real numbers by integers. Specifically, we were proving the following:

Proposition 1. *$\forall x \in \mathbb{R}, \exists N \in \mathbb{Z}$ and $\alpha \in [0, 1)$ s.t.*

$$x = N + \alpha.$$

Moreover, N, α are uniquely determined by x.

Proof. Last time we went through this in more depth, but here's where we were. We'll only consider $x \geq 1$. Let

$$J := \{n \in \mathbb{Z}_{\text{pos}} \ : \ n > x\}$$

be the set of positive integers greater than x. Since $\mathbb{Z}_{\text{pos}}$ is well ordered, J has a least element m. Let $N := m-1$ and $\alpha = x - N$. We proved last time that

(1) $N \in \mathbb{Z}_{\text{pos}}$
(2) $\alpha \in [0, 1)$
(3) $x = N + \alpha$ (by definition of α).

Thus, we've shown the existence of a solution, so let's now prove uniqueness, following Edith's proposal from last class. Suppose

$$x = N + \alpha = M + \beta,$$

where $N, M \in \mathbb{Z}_{\text{pos}}$ and $\alpha, \beta \in [0, 1)$. Without loss of generality (WLOG), let $M \geq N$. We have

$$M - N = \alpha - \beta.$$

Since M, N are positive integers with $M \geq N$, we get $M - N \in \mathbb{Z}_{\text{pos}} \cup \{0\}$ (proved in chapter 6 of the book). However, Emily noted $\alpha - \beta < 1$ since $\alpha < 1$ and $-\beta \leq 0$, so $\alpha - \beta < 1 - 0 = 1$. Since 1 is the least positive integer, we deduce that $M - N \notin \mathbb{Z}_{\text{pos}}$, whence $M - N = 0$. This in turn implies $\alpha - \beta = 0$. We've proved that $M = N$ and $\alpha = \beta$, which implies uniqueness! ☐

This proof seems fine, but it all relies on the well ordering of J, which is only true **as long as J is nonempty**. But we never proved that J is nonempty! We'll now prove a formal version of this, known as the **Archimedean Property of $\mathbb{R}$**.

2. Archimedean Property

Archimedean Property. *$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $n > x$. In other words, there are arbitrarily large positive integers.*

After combining pieces of proofs from Edith, Jenna, Sarah, Blakeley, Harry, and more, we arrived at the following:

Proof. Suppose, for the sake of contradiction, that there exists $x \in \mathbb{R}$ s.t. $x \geq n \ \forall n \in \mathbb{Z}_{\text{pos}}$. Then x is an upper bound on $\mathbb{Z}_{\text{pos}}$, so (A13) guarantees the existence of a least upper bound, i.e. the existence of some $\omega := \sup(\mathbb{Z}_{\text{pos}}) \in \mathbb{R}$. By definition, anything less than ω cannot be an upper bound on $\mathbb{Z}_{\text{pos}}$; in particular, $\omega - 1$ isn't an upper bound on $\mathbb{Z}_{\text{pos}}$. Thus, there's some $H \in \mathbb{Z}_{\text{pos}}$ s.t. $H > \omega - 1$. But then $H + 1 \in \mathbb{Z}_{\text{pos}}$ by closure of positives, and $H + 1 > \omega$, which contradicts that ω is an upper bound on the positive integers. ☐

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

Note that even though the statement of the Archimedean Property appears to use only concepts from (A1)-(A12), we used (A13) in the proof. This might lead you to wonder whether there might be a different proof that only relies on (A1)-(A12). It turns out there cannot be such a proof, since (as you'll see on your next problem set) there exist ordered fields that fail the Archimedean property!

Here's a corollary, also sometimes known as the Archimedean Property:

Proposition 2. *$\forall \epsilon > 0, \exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $\frac{1}{n} < \epsilon$. In other words, there are arbitrarily small positive numbers.*

Proof. (Suggested by Miles) Fix $\epsilon > 0$. By trichotomy, $\epsilon \neq 0$, so $\frac{1}{\epsilon} \in \mathbb{R}$. By the previous Archimedean Property, $\exists n > 1/\epsilon$, which rearranges to $\frac{1}{n} < \epsilon$. □

Remark. At first glance, this proof seems to go through when $\epsilon < 0$ as well... which is bad news, since the result should be false! Ben pointed out that we made crucial use of the positivity of ϵ in the step where we rearranged the inequality $n > \frac{1}{\epsilon}$.

Finally, here's a 3rd statement that's known as the Archimedean Property:

Proposition 3. *$\forall \epsilon > 0, \forall x \in \mathbb{R}, \exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $n\epsilon > x$.*

The proof is the similar to the one above.

3. Really Real Real Analysis

Time for our first real theorem. Recall that we previously proved (meta-analytically, at least) that $\sqrt{2}$ is not rational. However, we never proved that $\sqrt{2} \in \mathbb{R}$. This is the content of the following result:

Theorem 1. *$\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.*

Before writing down a rigorous proof, we do some scratchwork.

Scratchwork. *Intuitively, $\alpha = \pm\sqrt{2}$. Forrest reminded us that we already defined $\sqrt{2}$ in terms of suprema:*

$$\alpha := \sup \underbrace{\{x \in \mathbb{R} \; : \; x^2 \leq 2\}}_{\mathcal{A}}.$$

By (A13), we know $\alpha \in \mathbb{R}$. The problem is that we don't know, from this obscure definition, that $\alpha^2 = 2$!

How can we prove $\alpha^2 = 2$? We don't have any tools for verifying something like this. Actually, we do have one: trichotomy, which asserts that exactly one of $\alpha^2 = 2, \alpha^2 > 2, \alpha^2 < 2$ must hold. Our strategy is to show the latter two aren't possible, leaving what we want as the only option.

What goes wrong if $\alpha^2 > 2$? Intuitively, α should be slightly to the right of $\sqrt{2}$, which shouldn't happen since α is supposed to be the supremum of $\mathcal{A}$. In particular, if α is larger than $\sqrt{2}$, we expect that $(\alpha - \text{tiny})^2 > 2$... and this should give us a contradiction.

What do we mean by tiny? Well, we just saw one way to create arbitrarily small numbers, by considering fractions of the form $\frac{1}{n}$. Our goal is thus to find some $n \in \mathbb{Z}_{\text{pos}}$ s.t. $(\alpha - \frac{1}{n})^2 > 2$. This is equivalent to finding $n \in \mathbb{Z}_{\text{pos}}$ such that

$$\begin{aligned} \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} &> 2 \\ \alpha^2 - 2 &> \frac{2\alpha}{n} - \frac{1}{n^2} \\ \alpha^2 - 2 &= \frac{1}{n}\left(2\alpha - \frac{1}{n}\right), \end{aligned}$$

which means we win if we can find some n s.t.

$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha - \frac{1}{n}}. \tag{3.1}$$

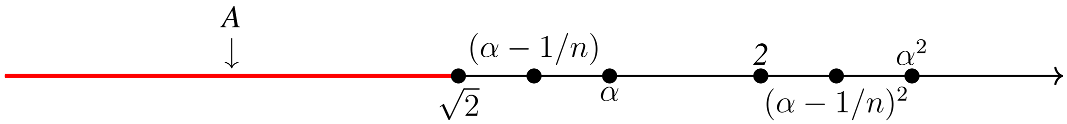
Now at this point it's tempting to invoke the Archimedean Property here, but we can't since the RHS depends on n. (Make sure you understand this point!) It's important to remember that we don't actually need to solve *the inequality* (3.1) *for* n*; we simply need to find a huge enough* n *that guarantees that* (3.1) *must hold. Jenna observed that if*

$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha}$$

this guarantees (3.1)*, since*

$$\frac{\alpha^2 - 2}{2\alpha} < \frac{\alpha^2 - 2}{2\alpha - \frac{1}{n}}$$

for any *positive integer* n*. Thus, we're in a position to apply Archimedean Property in a legit way and get to a contradiction! Before writing down a formal argument, here's a picture to stare at:*



Proof. Let $\mathcal{A} := \{x \in \mathbb{R} \ : \ x^2 \leq 2\}$, and define $\alpha := \sup \mathcal{A}$, which exists by (A13). Suppose $\alpha^2 > 2$. By Archimedean Property, $\exists n \in \mathbb{Z}_{\text{pos}}$ s.t.

$$n > \frac{2\alpha}{\alpha^2 - 2}.$$

It follows that

$$\begin{aligned} \frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha} &< \frac{\alpha^2 - 2}{2\alpha - \frac{1}{n}} \\ \implies \frac{1}{n}\left(2\alpha - \frac{1}{n}\right) &< \alpha^2 - 2 \\ \implies \left(\alpha - \frac{1}{n}\right)^2 &> 2, \end{aligned}$$

which implies $\alpha - 1/n$ is an upper bound of A. However, $\alpha - 1/n < \alpha$, which contradicts α being the least upper bound of A. Thus, $\alpha^2 \not> 2$.

It now remains only to show that $\alpha^2 \not< 2$, which we'll do next time! □