REAL ANALYSIS: LECTURE 9

OCTOBER 5TH, 2023

1. Real Real Analysis Theorem

Last time, we were in the process of proving the following:

Theorem 1. *$\exists \alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.*

Proof. Let
$$A := \{x \in \mathbb{R} \,:\, x^2 < 2\}$$
and set $\alpha := \sup A \in \mathbb{R}$, since its existence is guaranteed by the fact that $1 \in A$ and 27 is an upper bound on A. Here's a lemma:

Lemma 1. *If $\alpha^2 > 2$, then $\exists \epsilon > 0$ s.t. $(\alpha - \epsilon)^2 > 2$.*

Before proving this lemma, let's deduce $\alpha^2 \not> 2$ from assuming this claim. If $\alpha^2 > 2$, then by the above lemma $\exists \epsilon > 0$ s.t. $\alpha - \epsilon$ would be an upper bound on A, which contradict that $\alpha - \epsilon < \alpha$ is the *least* upper bound. Thus, $\alpha^2 \not> 2$ as long as we can prove this lemma, which we'll do now:

Proof. Suppose $\alpha^2 > 2$. By the Archimedean Property, $\exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $n > \frac{2\alpha}{\alpha^2 - 2}$, which implies
$$\frac{1}{n} < \frac{\alpha^2 - 2}{2\alpha} < \frac{\alpha^2 - 2}{2\alpha - 1/n}$$
$$\implies \frac{2\alpha}{n} - \frac{1}{n^2} < \alpha^2 - 2$$
$$\implies \left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > 2.$$
□

We can use an analogous idea on the other side:

Lemma 2. *If $\alpha^2 < 2$, then $\exists \epsilon > 0$ s.t. $(\alpha + \epsilon)^2 < 2$.*

After some scratchwork, a couple of proof attempts, and some work by Edith, Lexi, Emily, Sean, Aidan, Miles, Jon, Forrest, and more, we get the following:

Proof. Suppose $\alpha^2 < 2$. By Archimedean Property, pick $n \in \mathbb{Z}_{\text{pos}}$ s.t. $\frac{1}{n} < \frac{2-\alpha^2}{3\alpha}$. Recall $\alpha = \sup A$ and that $1 \in A$, which implies $\alpha \geq 1$. Thus, $\alpha \geq 1 \geq \frac{1}{m}\ \forall m \in \mathbb{Z}_{\text{pos}}$. Using this inequality we get
$$1/n < \frac{2-\alpha^2}{3\alpha} = \frac{2-\alpha^2}{2\alpha + \alpha} \leq \frac{2-\alpha^2}{2\alpha + 1/n}.$$
Simplifying as before, we find $\left(\alpha + \frac{1}{n}\right)^2 < 2$. Fixing $\epsilon = \frac{1}{n}$ we have found some $\epsilon > 0$ s.t. $(\alpha + \epsilon)^2 < 2$. □

[Note. The choice to use $\frac{2-\alpha^2}{3\alpha}$ was somewhat arbitrary; we could have easily used something else, for example $\frac{2-\alpha^2}{2\alpha+1}$, and the proof would have gone through just as well. The takehome lesson here is, in analysis, you don't have to find *the* right choice; any choice that makes the proof work is good enough.]

Given the above lemma, we are essentially done. Assume $\alpha^2 < 2$. By choosing ϵ as above, we get that $\alpha + \epsilon \in A$, which means that $\alpha + \epsilon > \alpha$ is *not* an upper bound. This contradicts α being the least *upper bound*. This is a contradiction that tells us $\alpha^2 \not< 2$. By trichotomy, it must be that $\alpha^2 = 2$, and we are done! □

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

One important lesson from our initially flawed attempts at proof is the following: if you know $a < b$ and $a < c$, you cannot deduce anything about the relationship between b vs. c! To make our proof work, we started with a known inequality $b < c$, and used the Archimedean Property to generate an inequality of the form $a < b$; this then forced $a < c$.

It turns out that our theorem, and the approach we took, generalize considerably (this is Theorem 7.5 in the book):

Theorem 2. $\forall n, k \in \mathbb{Z}_{\text{pos}}, \exists \sqrt[n]{k} \in \mathbb{R}$. *In other words, nth roots exist.*

Remark. The proof in the book is complicated—see the course website for Leo's take on the proof—but the ideas are all the same as what we just carried out, so make sure you understand the case of $\sqrt{2}$ before reading the proof of Theorem 7.5.

2. Rationals between Reals

Q: Given $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. We know there's a real number between them, but does there exist $q \in \mathbb{Q}$ s.t.

$$\alpha < q < \beta.$$

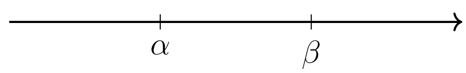
Notice you cannot simply average $\frac{\alpha+\beta}{2}$, since α, β might not be rational.

Proposition 1. *For any nonempty interval $(\alpha, \beta) \subseteq \mathbb{R}$,*

$$\mathbb{Q} \cap (\alpha, \beta) \neq \emptyset.$$

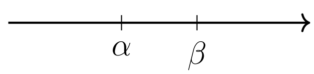
In other words, if $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$, there exists a rational number strictly between them.

Scratchwork. *Let's look at a picture.*

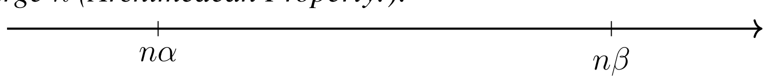


Annie noted that if $\alpha, \beta \in \mathbb{Q}$ this is easy! We can just take the midpoint, argue it's between, and argue it's a fraction. Lexi noted that if $\beta - \alpha > 1$, there should be some integer in (α, β). So, if there's a "long interval" we should be good.

Alex had the following idea: suppose $\beta - \alpha < 1$



Then multiply by some large n (Archimedean Property!):



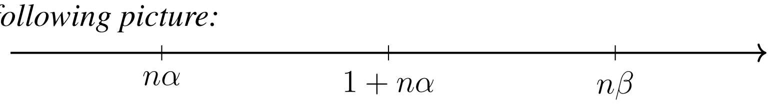
Now $n\beta - n\alpha > 1$, which means there should be some integer m in between $n\alpha$ and $n\beta$, which means that

$$n\alpha < m < n\beta \tag{2.1}$$

$$\alpha < \frac{m}{n} < \beta, \tag{2.2}$$

and we've found our rational number!

But this assumes we can find an integer m in the interval. This is certainly intuitive, but how can we actually do it? Well we have the following picture:



Ben observed that we have a way of creating an integer in this interval: simply take the floor of $n\beta$! Unfortunately, this doesn't necessarily produce an integer strictly inside the interval, since $n\beta$ might be an integer already. Instead, we follow [???]'s suggestion to take the floor of $1 + n\alpha$. Guided by the picture, we're now prepared for a formal argument:

Proof. Given $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha < \beta$. By the Archimedean Property, $\exists n \in \mathbb{Z}_{\text{pos}}$ s.t. $n > \frac{1}{\beta - \alpha}$. Thus,

$$n(\beta - \alpha) > 1 \implies n\beta > 1 + n\alpha.$$

Consider $\lfloor 1 + n\alpha \rfloor \in \mathbb{Z}$. We know

$$\lfloor 1 + n\alpha \rfloor \leq 1 + n\alpha < \lfloor 1 + n\alpha \rfloor + 1.$$

Notice that taking the right inequality and subtracting 1 gets us

$$n\alpha < \lfloor 1 + n\alpha \rfloor .$$

This, combined with the left inequality, tells us that

$$n\alpha < \lfloor 1 + n\alpha \rfloor \leq 1 + n\alpha.$$

Finally, $1 + n\alpha < n\beta$ (the interval is greater than 1). All together we get

$$n\alpha < \lfloor 1 + n\alpha \rfloor \leq 1 + n\alpha < n\beta,$$

which means that $\lfloor 1 + n\alpha \rfloor$ is in between $n\alpha$ and $n\beta$. It's also an integer! We deduce

$$\frac{\lfloor 1 + n\alpha \rfloor}{n} \in (\alpha, \beta) \cap \mathbb{Q}.$$

□

Note that this argument not only tells us existence of a rational in the interval, but tells us how to find one: for all large n, the fraction $\frac{\lfloor 1 + n\alpha \rfloor}{n}$ will lie in the interval!