# MATH 350: LECTURE 12

#### 1. Review

Last time, we gave the following definitions:

**Definition.**  $A \approx B$  if and only if  $\exists f : A \hookrightarrow B$  a bijection, i.e.  $\forall b \in B, \exists a \in A \text{ s.t. } f(a) = b$ .

**Definition.** We say  $f : A \hookrightarrow B$  is an **injection** iff f(a) = f(a') implies a = a'. (Informally,  $\forall b \in B$  there is at most one  $a \in A$  s.t. f(a) = b.)

**Definition.** We say  $f : A \twoheadrightarrow B$  is a surjection iff  $\forall b \in B \exists a \in A \text{ s.t. } f(a) = b$ . (Informally,  $\forall b \in B$ , there's at least one  $a \in A \text{ s.t. } f(a) = b$ .)

To prove that a function is a bijection, you need to prove that is both injective and surjective.

**Notation.** We use  $\hookrightarrow$  to denote an injection and  $\twoheadrightarrow$  to denote a surjection. We combine the two as  $\hookrightarrow$  to denote bijection.

#### 2. Comparing Sizes of Sets

Armed with these notions, we can define some familiar terms more rigorously.

**Definition.** We say a set A has n elements, denoted |A| = n, iff  $A \approx \{k \in \mathbb{Z}_{pos} : k \leq n\}$ . Also,  $|\emptyset| = 0$ .

**Definition.** A is <u>finite</u> iff  $\exists n \in \mathbb{Z}_{pos}$  s.t. |A| = n or  $A = \emptyset$ . A is <u>infinite</u> iff A is not finite.

Next we tried to formalize what it means for a set A to be at least as large as another set B. After several proposals and a lot of back and forth, we settled on the following

**Definition.** A is at least as large as B iff  $\exists f : B \hookrightarrow A$ .

Intuitively, if  $B \hookrightarrow A$ , then all the elements in B must map to different elements in A, which means there must be enough things in A to go around. Equivalently,  $A \twoheadrightarrow B$ , but we will almost always state things in terms of injections / bijections.

**Example 1.**  $\mathbb{Z}_{pos} \hookrightarrow [0,1]$  via the map  $n \mapsto \frac{1}{n}$ 

**Example 2.**  $(0,1) \hookrightarrow [0,1]$  via the map  $x \mapsto x$ 

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Last time, we had the intuition that  $[0,1] \approx (0,1)$ , so we should also be able to find an injection  $[0,1] \hookrightarrow (0,1)$ . Armie provided the mapping  $x \mapsto \frac{1}{2}x + \frac{1}{4}$ , which shrinks the closed interval down to half its size and shifts it over by 1/4. To check that this is in an injection, we assume there are  $x, y \in [1,0]$  which map to the same thing. We have

$$\frac{1}{2}x+\frac{1}{4}=\frac{1}{2}y+\frac{1}{4}\implies x=y$$

so this works! We might think intuitively that if you can find an injection between two sets in both directions, then they must be the same size. This turns out to be true, but it is surprisingly non-trivial to prove. It is a result known as the Cantor–Schröder–Bernstein theorem, because it was first proved by Dedekind.

**Theorem 2.1** (Cantor–Schröder–Bernstein). If  $\exists f : A \hookrightarrow B$  and  $\exists g : B \hookrightarrow A$ , then  $A \approx B$ .

You can read a full write up of this proof on the course website. This theorem is very useful in practice because it's usually much easier to find two separate injections than it is to find a single bijection (but you cannot use this on this week's problem set!).

Recall that we saw informally that  $\mathbb{Q}_{pos} \approx \mathbb{Z}_{pos}$ , but we gave a very handwavey argument that you could tabulate the rationals and then sweep diagonally across the table. It turns out that there is an explicit bijection  $\mathbb{Z}_{pos} \hookrightarrow \mathbb{Q}_{pos}$ . Consider

$$S(x) := \frac{1}{2\lfloor x \rfloor - x + 1}$$

For example, S(0) = 1, S(1) = 1/2, S(1/2) = 2, etc. It turns out that the sequence

$$S(0), \underbrace{S(S(0))}_{=:S^2(0)}, \underbrace{S(S(S(0)))}_{=:S^3(0)}, \dots$$

consists of each positive rational appearing exactly once, thus giving us a bijection between  $\mathbb{Z}_{pos}$  and  $\mathbb{Q}_{pos}$  via the map  $n \mapsto S^n(0)$ . A proof of this will be posted to the course website.

# 3. An Unexpected Tangent

Thomasina asked the following question: given A, B is it always the case that at least one of  $A \hookrightarrow B$  or  $B \hookrightarrow A$  is true? In other words, are any two sets comparable (using our notions of comparing sizes of sets)?

This turns out to be true if you accept what's called the **axiom of choice**, which is the assertion that, given an infinite number of sets, you can choose something from every set, i.e., you can create a new set which intersects all of your infinite sets in exactly one place. The axiom of choice might seem like it should be obviously true, but it has a bunch of bizarre consequences. One such result is the Banach-Tarski paradox: there exists a way to divide a solid ball of radius 1 into 5 pieces that you can take apart and put back together (without

changing the shapes of any of these 5 pieces) to get *two* solid balls, each of radius 1. Another consequence of Banach-Tarski: if you take a solid soccer ball, there exists a way of cutting it into finitely many pieces and then putting them back together to get a solid ball the size of the sun.

These results seem impossible, but it turns out the proof of Banach-Tarski is relatively straightforward, and the only questionable step in the proof is an invocation of the axiom of choice. It turns out that the statement in Thomasina's question (that there is always an injection between two sets) is equivalent to the axiom of choice.

#### 4. Game of 15

Problem 7.9 on the HW asserts  $\mathbb{R}$  is the unique complete ordered field, i.e. the only set satisfying axioms 1-13. Now, this is not literally true because we could, say, relabel  $\mathbb{R}$  in Hungarian. Hungarian  $\mathbb{R}$  is a little different because the elements have different names, but it is the same underlying set with the same underlying structure. But it would be silly to say that, from a mathematical perspective, this is a different set!

To illustrate this concept, we played a few rounds of the Game of 15. In the Game of 15, two players take turns choosing from the numbers 1 through 9, without replacement. The goal is to be the first person with exactly 3 numbers (any three) that sum to 15.

**Game 1: Leo vs. Students.** Leo starts with 5, Students follow with 7. Leo chooses 2, forcing Students to choose 8 to block him from getting 15. The game continues like this and ends in a draw.

$$L: 5, 2, 6, 3, 1$$
  
 $S: 7, 8, 4, 9$ 

Game 2: Leo vs. Alice. Alice goes first with 8, and the game proceeds in a similar manner, ending in a draw.

$$\mathbf{A}: 8, 6, 9, 4, 7$$
  
 $\mathbf{L}: 5, 1, 2, 3$ 

**Game 3. Leo vs Jackson.** Jackson starts with 5. Leo chooses 1. Jackson picks 2. Leo blocks with 8. Jackson blocks 6. But now Jackson can make 15 by choosing 4 (5 + 6 + 4) or 7 (2 + 6 + 7). Leo chooses 4, but Jackson wins by choosing 7.

**J** : 5, 
$$2, 6, 7$$
  
**L** : 1, 8, 4

Nathan observed that this game *feels* like tic-tac-toe, even though it doesn't *look* like tictac-toe. Denis added that we could turn it into tic-tac-toe by numbering a tic-tac-toe board with the digits 1 through 9. Jason pointed out we'd need the rows and columns to sum to 15. Here is a way to do this:

| 8 | 1 | 6 |
|---|---|---|
| 3 | 5 | 7 |
| 4 | 9 | 2 |

This is called a  $3 \times 3$  magic square: each row, column, and the two main diagonals sum to 15. Now it is not hard to see that the Game of 15 is really just tic-tac-toe on the magic square. For example:



Mathematicians say that the Game of 15 is **isomorphic** to tic-tac-toe. They are the same game, just played with different symbols. All of this is to say, what problem 7.9 is really saying is that  $\mathbb{R}$  is unique up to isomorphism! The concept of isomorphism is fundamental throughout mathematics.

# 5. Size

We took a brief digression to look at another way of thinking about the "size" of a set. For example, we saw that the interval (0,1) is uncountably infinite, but it is also natural to say (0,1) has *length* 1. We could imagine making this formal by, say, subtracting the supremum from the infinimum. Similarly, [0,1] has length 1. More generally, the length of any nonempty interval (a, b) has length b - a.

But how would we measure the length of a set that's not an interval? For example, what's the length of  $\mathbb{Q}$ ? We know  $\mathbb{R}$  is uncountable while  $\mathbb{Q}$  is countable, so we expect  $\mathbb{Q}$  to be minuscule compared to  $\mathbb{R}$ . In fact, we claim the length of  $\mathbb{Q}$  is 0. The idea of the proof is to cover all of  $\mathbb{Q}$  by intervals, the sum of whose lengths is tiny. Let's see how this works.

Proof idea. Since  $\mathbb{Q}$  is countable, we can write  $\mathbb{Q} = \{q_1, q_2, q_3, ...\}$ . (Make sure you understand this step, and why one can't make a similar claim about  $\mathbb{R}$ .) We will enclosed each of these points inside an open interval, as follows. First, create an open interval around  $q_1$  of length 1, e.g. the interval  $(q_1 - 1/2, q_1 + 1/2)$ . Next, create an open interval around  $q_2$  of length  $\frac{1}{2}$ , another around  $q_3$  of length  $\frac{1}{4}$ , and more generally, an open interval around  $q_n$  of length  $\frac{1}{2^{n-1}}$ .



The sum of the lengths of all of these intervals is

$$\leq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$$

(recall that this quantity is a "geometric series"). Note that we asserted that the sum of all the lengths of these intervals is  $\leq 2$  (as opposed to equal to 2) because some of these intervals might overlap.

Now observe that our choice to start with an interval of length 1 around  $q_1$  was arbitrary; we could have started with an interval of length  $\frac{1}{1000}$ , which would have led to the sum of all the lengths being  $\leq \frac{1}{500}$ . In this way, we see that we can cover all of  $\mathbb{Q}$  by intervals whose total length is arbitrarily small! Since we expect the length of anything to be nonnegative, and the only nonnegative number bounded above by arbitrarily small positive quantities is 0, we conclude that 0 is the only meaningful length we can assign  $\mathbb{Q}$ .