MATH 350: LECTURE 16

1. Review

Recall: $a_n \to L$ if $\forall \epsilon > 0$, $|a_n - L| < \epsilon$ for all large n. We saw last time that limits play nice with (A1) - (A12), that is, we can do ordinary algebra with them and they preserve order. We also proved the following proposition.

Proposition 1.1. If (a_n) converges, then (a_n) is bounded.

The idea here is that we know that for all large n, the a_n 's have to be within a distance ϵ from L, which immediately gives us an upper and lower bound for the tail of the sequence. Moreover, there are only finitely many n < N, so we can just take the maximum and minimum a_n 's here to get upper and lower bounds for the beginning of the sequence, and we conclude that the whole sequence is bounded.

2. Convergence Criteria

Since limits play nice with the first 12 axioms, we are prompted to ask whether they also play nice with (A13). But what exactly do we mean by this? Recall (A13) states that any non-empty subset of \mathbb{R} that's bounded above has a supremum in \mathbb{R} . Is there an analogue of this for sequences? For example, if (a_n) is bounded above, must it converge? Certainly not: just consider the decreasing sequence $a_n = -n$, which is bounded above by 0. Inspired by Proposition 1.1, we might try to impose a stronger condition: if (a_n) is bounded (i.e. bounded above and below), does (a_n) converge? The answer is still no, as Armie pointed out: $(-1)^n$ is bounded but does not converge. Thus we see that the clearest (albeit naive) analogy to (A13) fails. But maybe there are some other condition(s) we can impose on (a_n) to ensure convergence?

Divij proposed that if (a_n) is bounded and either strictly increases or stays the same, then we should expect it to converge.

Conjecture 2.1 (Divij). If (a_n) is bounded above and non-decreasing then (a_n) converges.

Definition. (a_n) is non-decreasing iff $a_{n+1} \ge a_n \forall n \in \mathbb{Z}_{pos}$. (Sometimes this is also called *increasing*.)

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Template by Leo Goldmakher.

So far, our only model for proving convergence is to start with a guess for the limit, so this is the approach we will use here.

PROOF IDEA. Given (a_n) non-decreasing and bounded above, say $a_n \leq M$. We claim

$$\lim_{n \to \infty} a_n = \underbrace{\sup\{a_n : n \in \mathbb{Z}_{pos}\}}_A$$

Note that a sequence itself does not have a supremum, but we take the supremum of the set of all the items of the sequence. We know the supremum exists since this set is nonempty (e.g., it contains a_1) and bounded above by assumption.

Given $\epsilon > 0$. We know that our sequence is getting closer to A from the left. Here's a picture:



Cameron observed that it's enough to show that the sequence eventually enters the interval $(A - \epsilon, A]$, because the conditions we impose on the sequence ensure it can't leave.

We first tried to proceed by contradiction, but formulating the proper statement of the negation proved quite challenging. While it's possible to do it this way, it turns out there's a more direct route using what we've already proved before (both in HW and on the midterm).

Proof of Divij's conjecture. Given $\epsilon > 0$. We claim

$$\lim_{n \to \infty} a_n = \underbrace{\sup\{a_n : n \in \mathbb{Z}_{pos}\}}_A.$$

We know $\exists a_N \in (A - \epsilon, A]$. Then, since (a_n) is increasing,

 $n \ge N \implies a_n \ge a_N > A - \epsilon$

On the other hand, A is an upper bound on the sequence, whence $a_n \leq A$. Putting this together, we deduce that $A - \epsilon < a_n \leq A$ for all large n. We conclude that

$$|a_n - A| = A - a_n < \epsilon. \qquad \Box$$

The same holds for (a_n) bounded below and non-increasing, and people often lump these two into one single result:

Theorem 2.2 (Monotone Convergence Theorem). If (a_n) is monotone and bounded, then (a_n) converges.

Definition. (a_n) is "monotone" iff it's non-decreasing *or* non-increasing.

Remark. MCT is our first <u>intrinsic</u> convergence criterion—you can use it to prove convergence without knowing the limit of the sequence in advance. By contrast, all our previous work with limits involved *conjecturing* a limit first, and then rigorously proving that our guess was correct.

Henry wondered whether MCT applies to sequences that are eventually monotone but which might have some alternating behavior at the beginning. Yes: you can prove this by lopping off the beginning of the sequence. But in this case, the limit will not in general be the supremum of the entire sequence, since it's possible that the first terms are wild and get really big before the sequence settles down into a tame increasing sequence.

To illustrate the utility of the MCT, we discussed a couple examples.

Example 1. $z_n := 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$.

This is clearly monotone, and on your HW you'll prove it's bounded. MCT therefore implies (z_n) converges. It turns out $z_n \to \frac{\pi^2}{6}$, which you probably wouldn't have been able to guess—demonstrating how useful it is to have an intrinsic criterion like the MCT! (The first person to guess what the limit was—and then prove it, too—was a remarkable mathematician named Euler.)

Remark. Jackson observed that the terms $\frac{1}{n^2}$ converge to zero. However, this is not enough to show that the sum z_n converges, as we'll soon discuss.

Example 2. $e_n := \left(1 + \frac{1}{n}\right)^n$.

This turns out to be bounded and increasing (good exercises!), so it converges by MCT. It turns out $e_n \rightarrow e_{\dots}$ which, once again, would have been very difficulty to guess!

Motivated by these examples, Cameron asked whether there's some intrinsic convergence criterion that doesn't assume monotonicity? After all, most sequences in the wild aren't monotone!

- Lily: we could try to construct a monotone subsequence of (a_n) , where a subsequence is a sequence $a_{n_1}, a_{n_2}, a_{n_3}, \cdots$ with $n_1 < n_2 < n_3 < \cdots$.
 - Denis raised an objection: it's not obvious we can actually find a monotone subsequence! For example, what if there are only finitely many increasing terms in the sequence?
- Max + Cameron: $\sup\{a_n : n \ge N\}$ and $\inf\{a_n : n \ge N\}$ should also converge to the limit of the sequence as N gets large

Taking Max and Cameron's idea and running with it, we observed a more basic phenomenon: any convergent sequence (a_n) eventually starts to clump together, i.e., the terms of the sequence get really close not only to the limit of the sequence but also to each other. Formally:

$$\forall \epsilon > 0, \ \exists N \in \mathbb{R} \text{ s.t. } m, n > N \implies |a_m - a_n| < \epsilon$$

Definition. We say a sequence (a_n) is "Cauchy" iff it satisfies the above condition.

It's pretty intuitively clear why any convergent sequence is Cauchy: eventually all the terms are close to the limit of the sequence, hence close to one another. (This isn't hard to turn into a formal proof; can you do it? We'll do it below, but it's a good exercise to stop and try on your own here.) What's much less clear is the converse: if a sequence starts to clump together does that guarantee convergence? Conceivably, the sequence might wiggle back and forth by some small amount and never quite converge. In fact, this never happens:

Theorem 2.3 (Cauchy Criterion). (a_n) is Cauchy iff (a_n) converges.

In other words, Cauchy-ness is *equivalent* to convergence. This is the type of intrinsic convergence criterion Cameron was asking for: the Cauchy criterion gives us a way of proving convergence without any foreknowledge of the limit *nor* any assumption of boundedness or monotonicity. In fact, because it's an equivalence, we can even use this to prove *divergence* of a sequence, which is something we weren't able to do with the MCT.

Example 3. $H_n := 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

Thus, for example, $H_1 = 1, H_2 = 3/2, H_3 = 11/6, \cdots$. The sum represented by H_n is called the *harmonic series*; we'll address this more carefully in later lectures. For now, we prove the following:

Claim. (H_n) diverges.

Proof. By the Cauchy criterion, it suffices to prove (H_n) isn't Cauchy. For any $k \in \mathbb{Z}_{pos}$,

$$\begin{aligned} H_{2k} - H_k &| = H_{2k} - H_k \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1} + \dots + \frac{1}{2k}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right) \\ &= \frac{1}{k+1} + \dots + \frac{1}{2k} \\ &\ge \underbrace{\frac{1}{2k} + \frac{1}{2k} + \dots + \frac{1}{2k}}_{k \text{ terms}} \\ &= \frac{1}{2} \end{aligned}$$

It follows that for any $N \in \mathbb{R}$, there exist m, n > N such that $|H_m - H_n| \ge \frac{1}{2}$. Thus (H_n) isn't Cauchy, hence doesn't converge.

Remark. The sequence (H_n) gives an example of a sum whose terms tend to 0, but which doesn't converge to anything. We will return to this subject in a future lecture.