REAL ANALYSIS: LECTURE 17

NOVEMBER 6$^{\text{th}}$, 2023

1. Preliminaries

Last time we proved the Monotone Convergence Theorem (MCT):

Theorem 1 (MCT)**.** *Given (a_n) monotone. Then (a_n) converges iff (a_n) is bounded.*

Recall that (a_n) is monotone iff it's either always increasing or always decreasing, where *increasing* means $a_{n+1} \geq a_n$ for all n. So for example the constant sequence $a_n = 3$ is monotone, since always increasing (it's also always decreasing!).

Although no mention of an actual limit appears in the statement of the MCT, secretly we know that the limit of (a_n) should be $\sup\{a_n\}$ or $\inf\{a_n\}$ (depending on whether a_n is increasing or decreasing). However, the MCT allows us to prove convergence without actually specifying the limit. This can be very useful!

Example 1. *Let*

$$a_n := 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2},$$

e.g. $a_1 = 1, a_2 = \frac{5}{4}, \ldots$. Notice $(a_n) \nearrow$ (convenient notation for saying that (a_n) is increasing). Thus, if (a_n) is bounded, MCT implies (a_n) converges. Now it's not clear how to prove that (a_n) is bounded, but at least we have a strategy that doesn't force us to determine a mysterious limit! (The limit turns out to be $\frac{\pi^2}{6}$, which is really not obvious and was discovered by Euler.)

The MCT gives us a way to tell if a sequence converges or not, without necessarily knowing the limit. However, it only applies to monotone sequences, a rather strong hypothesis.

Question 1. *Does there exist any intrinsic test for convergence for sequences that aren't monotone?*

Let's look at convergent sequences and try to look at their properties. Given (a_n) that converges, what can we say about (a_n)? Lexi suggests utilizing the Squeeze Theorem, which is a good idea but relies on creating sequences that depend on (a_n) itself. Jon suggests looking at subsequences of (a_n), which relates to a theorem in the book that all subsequences of (a_n) converges. Jenna describes that the terms of (a_n) will eventually clump together near the limit L.

Can we formalize this? Here's one idea. Jeremy suggests that the difference between subsequent terms gets small: $\forall \epsilon > 0$,

$$|a_{n+1} - a_n| < \epsilon$$

for all large n. But it's not just that subsequent terms clump next to each other. It's that *all* large terms clump very close to each other! In other words, $\forall \epsilon > 0$, $\exists N \in \mathbb{R}$ s.t.

$$|a_m - a_n| < \epsilon \qquad \forall m, n \geq N.$$

Any (a_n) satisfying this "clumping" is called a *Cauchy sequence* (first invented by Bolzano, later independently rediscovered by Cauchy). Here's an amazing theorem:

Theorem 2 (Cauchy Criterion)**.** *(a_n) converges iff (a_n) is Cauchy.*

Remark. If (a_n) converges, it's very believable that (a_n) is Cauchy. However, the other side of the implication is non-obvious—why can't the sequence jiggle back and forth forever without actually converging to a single fixed number?

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

We'll prove the Cauchy criterion next class. For now, let's look at some applications of it to appreciate its utility.

Example 2. *Let's go back to the example where*

$$a_n := \sum_{k \le n} \frac{1}{k^2},$$

i.e. $a_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$.

We claim (a_n) is Cauchy, i.e. that for all large m, n,

$$|a_m - a_n| < \text{tiny}.$$

WLOG $m \ge n$. Then

$$\begin{aligned} a_m - a_n &= \sum_{k=n+1}^{m} \frac{1}{k^2} \\ &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2}. \end{aligned}$$

Each of these terms are surely small, but there are also a lot ($m - n$) of terms we're adding! So, why is this small?

Here's a few attempts:

Attempt 1. *Here's an attempt by Gabe & Friends. Notice that*

$$\frac{1}{(n+k)^2} \le \frac{1}{(n+1)^2} \tag{1.1}$$

for any $k \ge 1$. Thus,

$$\begin{aligned} |a_m - a_n| &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(m-1)^2} + \frac{1}{m^2} \\ &< \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} \\ &= \frac{m-n}{(n+1)^2}. \end{aligned}$$

The problem with this is that the upper bound can get arbitrarily large as m grows. Note that this is not saying that (a_n) is not Cauchy; rather, it says that we "gave up too much" using Inequality 1.1.

Let's try again:

Attempt 2. *Here we're going to make each term slightly larger:*

$$\frac{1}{(n+k)^2} < \frac{1}{(n+k-1)(n+k)},$$

since we're making the denominator smaller. But notice we can write

$$\frac{1}{(n+k-1)(n+k)} = \frac{1}{n+k-1} - \frac{1}{n+k}.$$

Let's put this together:

$$\begin{aligned}
|a_m - a_n| &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(m-1)^2} + \frac{1}{m^2} \\
&< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(m-2)(m-1)} + \frac{1}{(m-1)m} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{m-2} - \frac{1}{m+1}\right) + \left(\frac{1}{m+1} - \frac{1}{m}\right) \\
&= \frac{1}{n} - \frac{1}{m} < \frac{1}{n}.
\end{aligned}$$

By Archimedean Property, we can thus make $|a_m - a_n|$ smaller than any positive ϵ. This idea of having the sum "telescope" and cancel each other out is known as a telescoping series.

Since the Cauchy criterion is an iff statement, we can also use it to prove that a sequence *diverges*. Here's an example:

Example 3. *Consider*

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

It turns out that H_n diverges. By the Cauchy criterion, it suffices to prove that there exist arbitrarily choices of m and n for which $|H_m - H_n|$ stays away from 0. Edith made this precise:

Proof. Observe that, for any n,

$$\begin{aligned}
|H_{2n} - H_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\
&> \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} \\
&= \frac{n}{2n} = \frac{1}{2}.
\end{aligned}$$

Thus, there exist arbitrarily large choices of m, n for which $|a_m - a_n|$ is large, and therefore (a_n) is not Cauchy. By the Cauchy criterion, (a_n) must diverge. □

Let's give a sketch of one last example which shows that we've all been secretly using the Cauchy criterion all our lives:

Example 4 (Infinite decimals are real numbers!)**.** *Suppose you have an infinite sequence of digits (d_n), i.e. for each n, $d_n \in \{0, 1, 2, \ldots, 9\}$. Consider the sequence (α_N) of longer and longer decimals:*

$$\alpha_N := 0.d_1 d_2 d_3 \cdots d_N.$$

Intuitively, we expect that α_N should converge to some real number, or in other words, that the sequence (α_N) converges. And it does! To see this, consider for example

$$|\alpha_5 - \alpha_{12}| = 0.00000 d_6 d_7 \cdots d_{12} < \frac{1}{10^5}.$$

Doing this with large m, n gets you something bounded below by $\frac{1}{10^n}$, which can get arbitrarily small. Thus, (α_n) is Cauchy and therefore converges.