MATH 350: LECTURE 18

1. Review

Last time we motivated and proved the following important result:

Theorem 1.1 (Cauchy Criterion). A sequence (a_n) is Cauchy iff (a_n) converges.

Remark. Keep in mind that the *definition of a Cauchy sequence* and the *Cauchy Criterion* are two very different things!

Q. Does the Cauchy criterion hold in every ordered field?

Noam observed that our proof of the Cauchy criterion used MCT, and our proof of MCT used (A13), so maybe we require (A13) for Cauchy criterion to hold. Armie gave us a potential counterexample: what if we have a sequence of rationals that converges to an irrational? to π , say?



The elements of the sequence get closer and closer to each other. In fact, we know it's Cauchy because it's Cauchy in \mathbb{R} . Yet it does not converge to anything in \mathbb{Q} ! Thus the Cauchy criterion fails in \mathbb{Q} .

But wait. Didn't we just say the sequence converges to π ? In \mathbb{R} it does, but if a sequence of rationals converges to an irrational, then in \mathbb{Q} , this sequence <u>diverges</u>. Recall that, by definition, a sequence diverges when there does not exist an element to which it converges.

Back to Noam's point: (A13) implies the Cauchy criterion. But it turns out that the Cauchy criterion also implies (A13). In other words, the Cauchy criterion is equivalent to the completeness axiom.

2. Redefining \mathbb{R}

In this course, we started by building \mathbb{R} , and then created other sets, like \mathbb{Z} and \mathbb{Q} , from \mathbb{R} . However, many real analysis courses go in the other direction: they start with the Peano axioms for \mathbb{Z}_{pos} , and then construct other types of numbers from \mathbb{Z}_{pos} . It's fairly straightforward to construct \mathbb{Z} from \mathbb{Z}_{pos} , and \mathbb{Q} from \mathbb{Z} . How would we go about constructing \mathbb{R} out of \mathbb{Q} ?

Date: November 11, 2024.

Template by Leo Goldmakher.

Evan: Define $\mathbb{R} := \{\lim_{n \to \infty} a_n : (a_n) \text{ is a sequence in } \mathbb{Q} \}.$

There were two objections to this definition:

- (i) (Thomasina) If the limit is not in \mathbb{Q} , and all we have is \mathbb{Q} , then how do we even define the limits or know what they are?
- (ii) (Cameron) What if $\lim_{n\to\infty} a_n$ doesn't converge to anything in \mathbb{R} ?

The latter issue is the easier to fix, so we start with that one: we restrict to Cauchy sequences (a_n) , which ensures the limit exists in \mathbb{R} . Issue (i), however, is much thornier. Dedekind resolved this by defining the elements of \mathbb{R} not as limits of sequences in \mathbb{Q} , but as the sequences themselves:

 $\mathbb{R} := \{ (a_n) \subseteq \mathbb{Q} : (a_n) \text{ is Cauchy} \}.$

For example, π is really just 3.1415965..., which is the sequence of rational numbers 3, 3.1, 3.14, 3.141, ... One immediate issue in this definition is: if we treat real numbers as sequences, how do addition, multiplication, inverses, etc work? For instance, what's the "sum" of two "numbers" in \mathbb{R} ? We can get around this issue by defining addition the natural way: the sum of two sequences is the sequence of sums, i.e.

$$(a_n) + (b_n) := (a_n + b_n).$$

We already showed that all the operations we know and love are preserved when we take limits, so this is okay. Similarly, the additive identity is the constant sequence $(0, 0, 0, \dots)$, and we can similarly make sense of all the other familiar properties in the language of sequences.

However, now we arrive at another problem: there may be more than one sequence converging to the same limit. For example, both $(0, 0, 0, \cdots)$ and $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$ converge to 0. But \mathbb{R} is supposed to have a *unique* additive identity! So which one of these do we pick? To eliminate this redudancy, Dedekind amended the definition above by declaring that two sequences are equivalent if they converge to the same value. Of course, we can't literally declare it this way, since we don't know what the value of the limit of a sequence is! Instead, we declare $(a_n) \sim (b_n)$ iff $(a_n - b_n)$ converges to 0, and then define

$$\mathbb{R} := \{(a_n) \subseteq \mathbb{Q} : (a_n) \text{ is Cauchy}\} / \sim .$$

If this notation is unfamiliar, it might seem very strange. But this essentially just means we declare (a_n) and (b_n) to be "the same" in our set if their difference converges to 0.

Thomasina posed some thoughtful philosophical commentary: isn't this all secretly circular because we are presupposing a rigorous understanding of \mathbb{R} ? This construction might *feel* circular to us, since it is informed and guided by our intuitions about \mathbb{R} ; logically, however, it does not make any assumptions about \mathbb{R} , so there is no circularity. We start from the definition of Cauchy, and from there we are able to recover familiar properties of \mathbb{R} . In fact,

we did a similar thing when we built \mathbb{R} out of the axioms: we used our knowledge of how \mathbb{R} should behave to come up with the axioms that uniquely define \mathbb{R} .

3. Series

Informally, a series is the sum of a sequence.

Example 1. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = 1.$

But what does this actually *mean*? How do we add infinitely many things together?? And if we never stop adding things, how do we ever get to 1???

Recall we originally defined addition as a *binary operation*: you add precisely two things. Then, by associativity, we saw that we can also add three things, and, it turns out, arbitrarily many things. But there is a critical distinction between adding *arbitrarily* many things and adding *infinitely* many things. (A1)-(A13) do not tell us how to add infinitely many things.

Well, we've seen one way of taking an infinity of things and assigning to it a meaningful finite value. Secretly, when we add infinitely many things we are taking the limit of a sequence. Here's what this actually means:

Definition. Given a sequence (a_n) . We say the series

$$\sum_{n=1}^{\infty} a_n = L$$

if and only if the sequence (S_N) of **partial sums** defined by

$$S_N := \sum_{n=1}^N a_n$$

converges to L.

Let's look at some concrete examples.

Claim.
$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Let's look at the first few partial sums.

$$S_{1} = \frac{1}{2}$$

$$S_{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_{3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Claim. $S_N = 1 - \frac{1}{2^N} \forall N \in \mathbb{Z}_{pos}.$

Proof. By induction.

Remark. Whenever you come up with conjecture about something "for all $n \in \mathbb{Z}_{pos}$ ", your instinct should be to use induction.

Now that we have a nice formula for the sequence of partial sums, we can prove by usual limit arguments that $S_N \to 1$. In fact, all these ideas generalize nicely:

Proposition 3.1. For any $\alpha \in (-1, 1)$,

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a}$$

This is called a *geometric series* with ratio a.

While exploring sequences, we've already encountered a few series:

Example 2. We proved $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ converges but did not prove the limit. **Example 3.** We proved $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges using Cauchy criterion. Keep in mind that evaluating $\sum_{n=1}^{\infty} a_n$ involves two different sequences: (a_n) and (S_N) .

One question that came up is how are they related, and what can we learn about one from the other? Suppose $\sum_{n=1}^{\infty} a_n$ converges. What does this tell us about (a_n) ?

Proposition 3.2. (Max) If sum (a_n) converges, then $a_n \rightarrow 0$.

Intuition. If the elements aren't going to zero, then we will keep adding non-small things to our sum, so it will keep changing and never converge.

Proof. Note that

$$S_N - S_{N-1} = a_1 + a_2 + \dots + a_{N-1} + a_N - (a_1 + a_2 + \dots + a_{N-1}) = a_N$$

Then

$$(S_N)$$
 converges $\implies (S_N)$ is Cauchy $\implies a_N = S_N - S_{N-1} \longrightarrow 0.$

Is the converse to Max's proposition true? Absolutely not! In fact, we've already seen a counterexample: the sequence $\frac{1}{n} \to 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.