MATH 350: LECTURE 19

1. Review

Recall we began talking about **series**. We said that

$$\sum_{n=1}^{\infty} a_n = L \quad \text{if and only if} \quad S_N \longrightarrow L,$$

where the partial sums S_N are defined by

$$S_N := \sum_{n=1}^{\infty} a_n.$$

If there exists such an $L \in \mathbb{R}$, we say $\sum_{n=1}^{\infty} a_n$ converges; otherwise, it diverges.

So far, we've proved:

2. Alternating Harmonic Series

By contrast to the Harmonic series, we define the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

Let's write out the first several terms to get a better idea of what's going on:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$$

Most of us agreed that this series converges. Daniel gave the following argument:

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Template by Leo Goldmakher.

Observe that $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$, which gives us an upper bound on each pair of terms:

$$\underbrace{1-\frac{1}{2}}_{<1} + \underbrace{\frac{1}{3}-\frac{1}{4}}_{<\frac{1}{3^2}} + \underbrace{\frac{1}{5}-\frac{1}{6}}_{<\frac{1}{5^2}} + \dots < 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dotsb$$

If we group the terms by pairs and sum over these pairs, we get something that is bounded above by $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which we proved converges. This gives us an upper bound on the subsequence (S_{2N}) . Moreover, we know (S_{2N}) is monotonically increasing since the sum of each pair is positive. Hence MCT implies that (S_{2N}) converges.

Let's prove this rigorously.

Proof.
$$\forall n \in \mathbb{Z}$$
,

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}.$$

This implies

$$S_{2N} < 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2N-1)^2} \le \sum_{n=1}^{2N-1} \frac{1}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2},$$

so (S_{2N}) is bounded above. Also, (S_{2N}) is increasing because $\forall N \in \mathbb{Z}_{pos}$,

$$S_{2(N+1)} = S_{2N} + \frac{1}{2N+1} - \frac{1}{2N+2} = S_{2N} + \frac{1}{(2N+1)(2N+2)} > S_{2N}$$

Thus (S_{2N}) converges by MCT, say $S_{2N} \longrightarrow E$.

Let's step back from the trees for a moment and try to see the forest. We're trying to prove (S_N) converges, but nowhere have we said it suffices to show that (S_{2N}) converges, so why are we even looking at (S_{2N}) at all? Daniel made a nice observation that gave us an easy path to show the convergence of (S_{2N}) , which relied on its monotonicity. But (S_N) is clearly not monotone, so the path to convergence was not as clear. How might we now be able to use the convergence of (S_{2N}) to show that (S_N) converges?

Observe that $S_{2N-1} - S_{2N} = \frac{1}{2N}$ is small, so consecutive terms seem to be clumping. Why does this mean S_N must converge? Noam gave as an idea: we know (S_{2N}) gets close to E, and we know that consecutive terms get close, which means we should expect the other terms of (S_N) to get close to E.

Claim. $S_\ell \to E$

<u>Idea</u>: For large, even ℓ we know

$$S_\ell \approx E$$

For large, odd ℓ , we know

$$S_\ell \approx S_{\ell+1} \approx E.$$

You know what this means: triangle inequality!

Proof. Given $\epsilon > 0$. Then $\exists N$ s.t.

$$\ell > N \implies |S_{2\ell} - E| < \frac{\epsilon}{2}$$

and

$$\ell > \frac{1}{\epsilon} \implies |S_{2\ell-1} - S_{2\ell}| = \frac{1}{2\ell} < \frac{\epsilon}{2}.$$

(Armie asked: doesn't the above argument show that the sequence is Cauchy? But recall that in a Cauchy sequence, we need any two terms in the sequence to get close, not just consecutive terms!)

Now $\forall D > \max\{2N, \frac{1}{\epsilon}\}$, we have two cases:

If D is even,

$$|S_D - E| < \epsilon.$$

If D is odd,

$$|S_D - E| = |S_D - S_{D+1} + S_{D+1} - E| \le |S_D - S_{D+1}| + |S_{D+1} - E| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $S_D \longrightarrow E$, whence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Nathan wondered what the series converges to. Dirichlet discovered a neat trick for computing this sum. Define

$$F(x) := x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = F(1).$$

and note that

$$F'(x) = 1 - x + x^2 - x^3 + x^4 + \dots = \frac{1}{1+x},$$

so we have

$$F(1) = F(1) - F(0) = \int_0^1 F'(x) dx = \int_0^1 \frac{1}{1+x} dx = \log(1+x) \Big|_0^1 = \log 2.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$$

But Dirichlet discovered some other, rather unsettling things about this sum. Let's rearrange the terms in order to sum them more conveniently:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

$$= \underbrace{1 - \frac{1}{2} - \frac{1}{4}}_{\frac{1}{2} - \frac{1}{4}} + \underbrace{\frac{1}{3} - \frac{1}{6} - \frac{1}{8}}_{\frac{1}{6} - \frac{1}{8}} + \underbrace{\frac{1}{5} - \frac{1}{10} - \frac{1}{12}}_{\frac{1}{10} - \frac{1}{12}} + \underbrace{\frac{1}{7} - \frac{1}{14} - \frac{1}{16}}_{\frac{1}{14} - \frac{1}{16}} + \cdots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots$$

$$= \frac{1}{2} (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots)$$

Thus we've proved that the sum equals half of itself, whence

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = 0.$$

Huh? Didn't we just say the sum was $\log 2$?

Dirichlet realized that something fishy was going on. Somehow rearranging the terms of an infinite series changes the sum. Evidently, the commutativity of ordinary addition is not a property enjoyed by infinite sums. By the way, this should convince you of the fact that a series is not literally a sum of infinitely many terms!

However, Dirichlet figured out the condition that guarantees you can rearrange the terms without changing the convergence.

Theorem 2.1 (Dirichlet). Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. Then any rearrangement of $\sum_{n=1}^{\infty} a_n$ converges to the same value.

This leaves a natural question: what happens if we weaken the hypothesis? A couple decades later, Riemann proved that Dirichlet's hypothesis was exactly the right one:

Theorem 2.2 (Riemann). If
$$\sum_{n=1}^{\infty} a_n$$
 converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\forall \alpha \in \mathbb{R}$, there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to α .