#### MATH 350: LECTURE 20

# 1. Series Convergence

Last time, we answered the following question: given  $\sum_{n=1}^{\infty} a_n$  that converges, must any rearrangement also converge to the same value?

As we saw, the answer in general is no. Dirichlet computed

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$$

but found a rearrangement of the series which converged to 0. But what exactly do we mean by *rearrangement*?

**Definition.** A rearrangement of 
$$\sum_{n=1}^{\infty} a_n$$
 is  $\sum_{n=1}^{\infty} a_{f(n)}$  where  $f : \mathbb{Z}_{pos} \hookrightarrow \mathbb{Z}_{pos}$ .

With this formal definition, let's revisit the theorems we stated last time.

**Theorem 1.1** (Dirichlet). If  $\sum_{n=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges, and every rearrangement of  $\sum_{n=1}^{\infty} a_n$  converges to the same value.

Dirichlet's theorem doesn't tell us anything about what happens when  $\sum_{n=1}^{\infty} |a_n|$  doesn't converge. But Riemann proved a complement about 20 years later:

**Theorem 1.2** (Riemann). If 
$$\sum_{n=1}^{\infty} a_n$$
 converges but  $\sum_{n=1}^{\infty} |a_n|$  diverges, then  $\forall \alpha \in \mathbb{R} \cup \{\pm \infty\}$   
there exists  $f : \mathbb{Z}_{pos} \hookrightarrow \mathbb{Z}_{pos}$  such that  $\sum_{n=1}^{\infty} a_{f(n)} = \alpha$ .

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Template by Leo Goldmakher.

These two theorems reveal a striking dichotomy of convergent series: if the series of absolute values converges, then the series convergence is nice and immune to rearrangement. If it diverges, all hell breaks loose.

Because this condition on  $\sum_{n=1}^{\infty} |a_n|$  completely characterizes the behavior of convergent series,

it deserves a name.

**Definition.** We say 
$$\sum_{n=1}^{\infty} a_n$$
 converges absolutely iff  $\sum_{n=1}^{\infty} |a_n|$  converges. We say  $\sum_{n=1}^{\infty} a_n$  converges conditionally iff  $\sum_{n=1}^{\infty} a_n$  converges but not absolutely.

It turns out this behavior can be generalized for series in  $\mathbb{C}$ : the set of all rearrangements of a convergent series is either a single point, a line in the complex plane, or the entire complex plane. This holds even more generally in vector spaces: the set of all rearrangements of a convergent series always spans some subspace. This is called the Lévy-Steinitz theorem.

# 2. Alternating Series Test

Last time, we proved that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n}$$

converges. Here was the big idea:

Daniel observed that a subsequence  $(S_{2N})$  of the partial sums is monotone and bounded and hence, by MCT, converges, say  $S_{2N} \longrightarrow A$ . But how did we use this to show  $(S_N)$  converges? We observed that  $|S_{2N-1} - S_{2N}| = \frac{1}{2N}$  is small. Therefore,  $S_{2N-1} \approx S_{2N} \approx A$ .

Keel put this intuition into words: if you have some subsequence that you know converges, and you can prove that any terms not in the subsequence get close to terms in the subsequence, then you can prove that the sequence converges. This idea gives rise to a convergence test that might be familiar from calculus:

**Theorem 2.1** (Alternating Series Test). Given  $(a_n)$  monotonically decreasing with  $a_n \to 0$ . Then  $\sum_{n=1}^{\infty} (-1)^{n+1}(a_n)$  converges.

Here is a sketch of the proof:

### Proof sketch.

(1)  $(S_{2N})$  is monotonically increasing (group by pairs:  $(a_1 - a_2) + (a_3 - a_4) + \cdots$ )

- (2)  $S_{2N-1}$  bounded above by  $a_1$  (group by pairs differently:  $a_1 (a_2 a_3) (a_4 a_5) \cdots$ )
- (3)  $S_{2N} \approx S_{2N-1} \forall$  large N, so conclude the proof using the  $\epsilon$  definition of limit convergence.

Making this precise is an excellent exercise. (It's also in the book!)  $\Box$ 

This concludes our study of series. Now onto...limits?

# 3. Limits of Functions

We know what it means for a sequence to have a limit, but what about limits of functions? For example, what does

$$\lim_{x \to 2} f(x) = 5$$

actually mean?

Wyatt suggested: As x gets close to 2, f(x) gets close to 5. How can we make this rigorous? Maybe our function looks something like this:



Drawing on our knowledge of limits of sequences, Nathan started us off with a first draft:

**Definition** (1.0).  $\lim_{x\to 2} f(x) = 5$  if and only if  $\forall \epsilon > 0 \exists N$  such that

$$n > N \implies |f(2+\frac{1}{n})-5| < \epsilon \text{ and } |f(2-\frac{1}{n})-5| < \epsilon.$$

Max pointed out this definition only monitors the behavior of the function at a discrete set of points. But our function might behave wildly in between these points, which is not so great. Evan proposed we fix this by just letting  $\frac{1}{n}$  be a real number, say y. Then, instead of n > N for some large N, we need  $y < \delta$  for some small  $\delta$ . Further, we noticed that we can compress the two inequalities into one by by simply imposing  $|y| < \delta$ . Putting this altogher, we arrived at a second draft:

**Definition** (1.5).  $\lim_{x \to 2} f(x) = 5$  if and only if  $\forall \epsilon > 0 \ \exists \delta > 0$  such that  $|y| < \delta \implies |f(2+y) - 5| < \epsilon.$ 

This definition is better, but we can clean it up a little bit. Since the limit is a statement about the behavior of f(x) "as x approaches 2", we want our definition to be in terms of x, instead of introducing a new variable y. After all, what are we really saying? Given some tiny<sub>1</sub> distance, we need to be able to find some other tiny<sub>2</sub> distance such that whenever x is within tiny<sub>2</sub> distance from 2, f(x) is within tiny<sub>1</sub> distance from 5. Here's a picture:



Substituting  $2 + y \mapsto x$ , our definition above becomes:

**Definition** (2.0).  $\lim_{x \to 2} f(x) = 5$  if and only if  $\forall \epsilon > 0 \exists \delta > 0$  such that  $|x - 2| < \delta \implies |f(x) - 5| < \epsilon.$ 

This is starting to look pretty good. But we noticed still more issues:

- (i) (Armie) What if x isn't in the domain of f?
- (ii) (Nathan) What if  $f(2) \neq 5$ ?

Let's address Nathan's issue first. Perhaps our function does something like this:



Certainly x = 2 is close to 2, yet f(2) is not close to 5. But we know from experience that the limit should still be 5, so the above definition breaks. This highlights a fundamental observation about limits:

Limits describe the behavior of a function *near* a point, not *at* a point.

We can therefore fix our definition by simply excluding x = 2. Then our definition above generalizes as follows:

**Definition** (3.0).  $\lim_{x \to a} f(x) = L$  if and only if  $\forall \epsilon > 0 \ \exists \delta > 0$  such that  $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$ 

This is essentially correct, but there are a couple of technical points left to deal with. We'll start next class by addressing Armie's issue next lecture.