REAL ANALYSIS: LECTURE 21

NOVEMBER 27TH, 2023

1. PRELIMINARIES

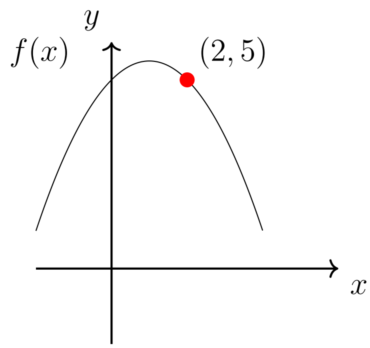
Last time we talked about *metric spaces*, which serves as a proof of concept that we can deal with sequences and limits with only a set equipped with a few slight properties. Now we're going to move on and begin talking about *limits of functions*.

1.1. Limits of Functions.

Here's the key idea. What does the following mean?

$$\lim_{x\to 2} f(x) = 5.$$

Intuition 1. *Here's the informal intuition. As you plug in values closer and closer to* 2*, the function* f *outputs values closer and closer to* 5*. Picture time!*



Cool! Let's ϵ the hell out of this:

Definition (Limit- Wrong Definition!)**.**

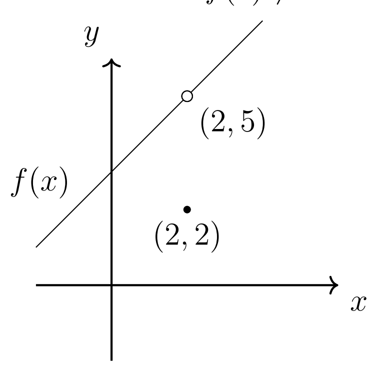
$$\lim_{x\to 2} f(x) = 5$$

iff $\forall\epsilon > 0, \exists\delta > 0$ s.t.

$$|x-2| < \delta \implies |f(x)-5| < \epsilon.$$

Intuition 2. *Here's the definition in English. Fix a tolerance* $\epsilon > 0$*. The limit existing means if you fix this tolerance I can find enough room (specifically* δ *room) such that if* x *is within* δ *of* 2 *(i.e.* $|x-2| < \delta$*) then* $f(x)$ *is within* ϵ *of* 5 *(i.e.* $|f(x)-5| < \epsilon$*).*

Here Lexi brought up a great point. What about when $f(2) \neq 5$. Here's the picture:



Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

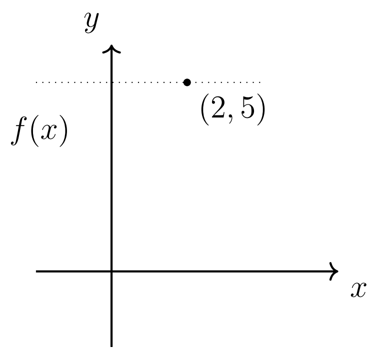
Intuitively, the limit should still be 5. But does this follow from our formal definition? No! $f(2) \neq 5$ will prevent our definition from working. Let's fix this by just not allowing $x = 2$:

Definition (Limit- Also Wrong Definition!)**.** $\lim_{x\to 2} f(x) = 5$ iff $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$0 < |x-2| < \delta \implies |f(x) - 5| < \epsilon.$$

Here the $0 < |x-2|$ prevents $x = 2$. But, this actually doesn't quite work as well. Consider the following:

$$\begin{aligned} f :& \mathbb{Q} \to \mathbb{R} \\ & x \mapsto 5. \end{aligned}$$



Ok yes technically since $\mathbb{Q}$ dense in $\mathbb{R}$ this should actually just look like $f(x) = 5$ (i.e. although there are gaps in $\mathbb{Q}$ we can't visibly see them), but this is the general idea. Intuitively, the limit (as x approaches anything!) should be 5, but our current definition will fail on the technicality that $f(x)$ wouldn't even be well defined (what's $f(\sqrt{2})$? Well, it doesn't even make sense to ask that question!). Let's just fix this by adding in we can only plug in stuff from the domain. Ok, take 3:

Definition (Limit- Also Also Wrong Definition!)**.** Given $f : X \to \mathbb{R}$. We say

$$\lim_{x\to a} f(x) = L$$

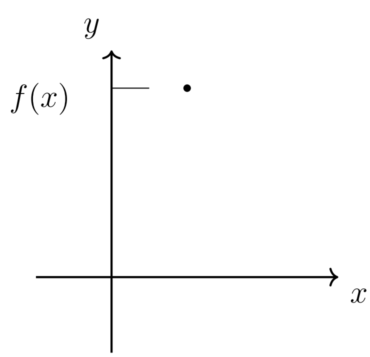
iff $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$0 < |x-a| < \delta \text{ and } x \in X \implies |f(x) - L| < \epsilon.$$

Hmm. Here's a really annoying function.

$$\begin{aligned} f :& [0,1] \cup \{2\} \to \mathbb{R} \\ & x \mapsto 5. \end{aligned}$$

Here's what this looks like:



We now can (unfortunately) prove the following:

$$\lim_{x\to 2} f(x) = \pi.$$

Proof. Fix $\epsilon > 0$. Then $\forall x \in [0,1] \cup \{2\}$ s.t. $0 < |x-2| < \frac{1}{2}$, we have that $|f(x) - \pi| < \epsilon$. Why? Because there are no such x's! There is no x that simultaneously is in the domain and also is within $\delta = 1/2$ of 2, which means that our claim $|f(x) - \pi| < \epsilon$ is vacuously true. $\square$

Ugh. Ok, so take 3 didn't work. The problem here is that 2 is an *isolated point*. In other words, $\exists\delta > 0$ s.t.

$$(2-\delta, 2+\delta) \cap \text{Domain}(f) = \{2\}.$$

Whenever this happens our definition will fail. To fix this we will restrict it so that the a's we'll look at must be *accumulation points*. First let's define this:

Definition (Accumulation Point)**.** Given $X \subseteq \mathbb{R}$, we say a is an *accumulation point of* X iff $\forall\delta > 0$,

$$((a-\delta, a+\delta) \setminus \{a\}) \cap X \neq \emptyset.$$

Really what this means is that no matter how close you look (within δ), you will always see stuff besides just a. Here's an example:

Example 1. Consider $(0, 1]$. Is 1 an accumulation point? Yes! Fix any $\delta > 0$. Then there's definitely other stuff besides 1 in $(1-\delta, 1+\delta)$. Specifically, when you move to the left ($1-\delta$ part), there's still stuff. Same with 0! So, an accumulation point doesn't need to live in X, but there must be points super close by.

Ok, correct definition time:

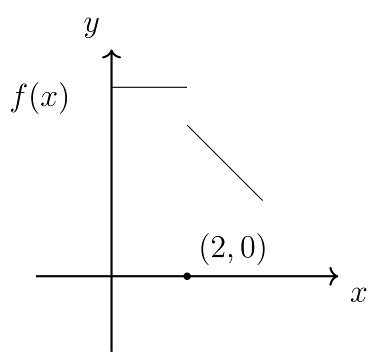
Definition (Limit)**.** Given $f : X \to \mathbb{R}$ with $X \subseteq \mathbb{R}$, we say

$$\lim_{x\to a} f(x) = L$$

iff a is an accumulation point of X and $\forall\epsilon > 0, \exists\delta > 0$ s.t.

$$(0 < |x-a| < \delta) \wedge (x \in X) \implies |f(x) - L| < \epsilon.$$

What about this function though?



Shouldn't this have two limits as $x \to 2$? Well, no. If we look within ϵ of $x = 2$, we note that there's lots of room between $f(x+\epsilon)$ and $f(x-\epsilon)$. Intuitively there's too much room here, which will prevent us from proving there's a limit at all. Here's the point. We will not directly enforce that there must be a unique limit. However, a consequence of our precise definition is that if a limit does exist, it will be unique (proof of this is analogous to that of sequences). With this formal definition, let's look at some examples.

1.2. **Examples.**

Example 2. Consider

$$\begin{aligned} f :&\mathbb{R} \to \mathbb{R} \\ &x \mapsto x^2. \end{aligned}$$

We claim $\lim_{x\to 2} f(x) = 4$. Let's do some scratchwork to build up a formal proof.

Scratchwork 1. *Since $f : \mathbb{R} \to \mathbb{R}$, we don't need to worry about accumulation points (we can plug in anything!), so we just need to worry about the $\epsilon - \delta$ stuff. Given $\epsilon > 0$, we need to find an $\delta > 0$ s.t.*

$$0 < |x-2| < \delta \implies |x^2 - 4| < \epsilon.$$

Ok, let's find some δ (in terms of ϵ) that works. Well,

$$|x^2 - 4| < \epsilon \iff |x-2||x+2| < \epsilon.$$

We also know that if $x \approx 2, x+2 \approx 4$. Hmm. So maybe we're going to get a factor of 4 somewhere, which leads us to the initial guess of $\delta = \epsilon/4$. Let's see if this works:

Ok, formal proof time.

Proof. THIS PROOF DOESN'T WORK! Given $\epsilon > 0$. Then $\forall x$ s.t. $0 < |x-2| < \frac{\epsilon}{4}$, we have that

$$-\epsilon/4 < x - 2 < \epsilon/4$$
$$4 - \epsilon/4 < x + 2 < 4 + \epsilon/4.$$

Well since ϵ tiny we know that $4 - \epsilon/4 > 0$, which means $x + 2 > 0$. So,

$$x + 2 = |x+2| < 4 + \epsilon/4 < 5.$$

Thus,

$$\begin{aligned} |x^2 - 4| &= |x-2||x+2| \\ &< \frac{\epsilon}{4} \cdot 5 = \frac{5}{4}\epsilon. \end{aligned}$$

Hmm, so this is slightly off. Let's just change δ to make it work. □

Proof. THIS PROOF ALSO DOESN'T WORK! Given $\epsilon > 0$. Then $\forall x$ s.t. $0 < |x-2| < \frac{\epsilon}{5}$, we have that

$$-\epsilon/5 < x - 2 < \epsilon/5$$
$$4 - \epsilon/5 < x + 2 < 4 + \epsilon/5.$$

Well since ϵ tiny we know that $4 - \epsilon/5 > 0$, which means $x + 2 > 0$. So,

$$x + 2 = |x+2| < 4 + \epsilon/5 < 5.$$

Thus,

$$\begin{aligned} |x^2 - 4| &= |x-2||x+2| \\ &< \frac{\epsilon}{5} \cdot 5 = \epsilon, \end{aligned}$$

and we're done! □

There's still a problem here, pointed out by Gabe. We're assuming here that ϵ is tiny, but that's not necessarily the case. What if $\epsilon = 100$. Then $4 + \epsilon/5$ is definitely not < 5. Notice that when ϵ is huge this should be really easy to prove, but still we can't assume it. Let's do a cheap trick to make this work:

Proof. Given $\epsilon > 0$. $\forall x$ s.t. $0 < |x-2| < \min\{1, \epsilon/5\}$, we have

$$-1 < x - 2 < 1$$
$$3 < x + 2 < 5,$$

so $|x+2| < 5$, which implies

$$\begin{aligned} |x^2 - 4| &= |x-2||x+2| \\ &< \frac{\epsilon}{5} \cdot 5 = \epsilon. \end{aligned}$$

We can claim $|x-2| < \epsilon/5$ because it's smaller than the *minimum* of $\epsilon/5$ and 1, which means it's certainly less that $\epsilon/5$. □

The cheap trick here is this. If you choose a really high tolerance ϵ, we'll just use 1 instead of $\epsilon/5$.

Let's do another example:

Example 3. Consider

$$\begin{aligned} f :&\mathbb{R} \to \mathbb{R} \\ x \mapsto &\begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0. \end{cases} \end{aligned}$$

What is $\lim_{x\to 0} f(x)$? Well, what does this look like. When $x > 0$ we have $f(x) = x/x = 1$, and when $x < 0$ we have $f(x) = -x/x = -1$. So, this is a piecewise function, which intuitively means the limit shouldn't exist. Let's prove this.

Proof. Suppose $\lim_{x\to 0} f(x) = L$. Then $\exists \delta > 0$ s.t.

$$0 < |x - 0| < \delta \implies |f(x) - L| < \frac{1}{10}.$$

The problem here is that no L is simultaneously close to 1 and -1. Formally, we get that

$$\begin{aligned} \left| f\left(\frac{-\delta}{2}\right) - L \right| &< \frac{1}{10} \\ \left| f\left(\frac{\delta}{2}\right) - L \right| &< \frac{1}{10}, \end{aligned}$$

which means

$$\begin{aligned} |-1 - L| &< \frac{1}{10} \\ |1 - L| &< \frac{1}{10}. \end{aligned}$$

Let's use triangle inequality to show this can't be true. Note $|-1-L| = |-1(1+L)| = |1+L|$, and that

$$\begin{aligned} 2 &= |1 + L + 1 - L| \\ &\leq |1 + L| + |1 - L| \\ &< \frac{1}{10} + \frac{1}{10} = \frac{1}{5}, \end{aligned}$$

a contradiction. □