MATH 350: LECTURE 22

1. LIMITS OF FUNCTIONS

Recall we defined the limit of a function:

Definition. Given $f: X \subseteq \mathbb{R} \to \mathbb{R}$ and a an accumulation point of X. Then $\lim_{x \to a} f(x) = L$ if and only if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t.

$$0 < |x - a| < \delta$$
 and $x \in X \implies |f(x) - L| < \epsilon$.

Henry reminded us that an accumulation point is a point you can "zoom in" on as much as you want and you will always see points from X. Recall that accumulation points need not be in the domain of the function.

Question. What is the connection between limits of sequences and limits of functions? Max proposed the following:

Proposition 1.1. Given $f : \mathbb{R} \to \mathbb{R}$. If $\lim_{x \to a} f(x) = L$, then $\forall (s_n) \ s.t. \ s_n \to a, \ f(s_n) \to L$.

Is this an if and only if? That is, if $\forall (s_n)$ s.t. $s_n \to a$, $f(s_n) \to L$, does this force $\lim_{x\to a} f(x) = L$?

Some debate ensued. Max realized we could have a constant sequence (a, a, a, ...) which clearly converges to a, but maybe $f(a) \neq L$. We amended our proposition to require that $s_n \neq a \forall n$, i.e. $a \notin \{s_n : n \in \mathbb{Z}_{pos}\}$.

More discussion ensued. Ultimately we arrived at the following:

Proposition 1.2. Given $f : \mathbb{R} \to \mathbb{R}$. Then $\lim_{x \to a} f(x) = L$ if and only if $\forall (s_n) \ s.t. \ s_n \to a$ and $s_n \neq a$ for any n, $\lim_{n \to \infty} f(s_n) = L$.

This is proved in the book, but we outlined a sketch:

(Proof sketch).

 (\Rightarrow) Given $\lim_{x \to a} f(x) = L$. Then whenever $x \approx a$, $f(x) \approx L$. If $s_n \to a$, then $s_n \approx a$ fo all large n, hence $f(s_n) \approx L$ for all large n.

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By definition, $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exists some $x \in (a - \delta, a) \cup (a, a + \delta)$ satisfying

$$|f(x) - L| \ge \epsilon.$$

Then

$$\exists x_1 \text{ s.t. } 0 < |x_1 - a| < 1 \text{ and } |f(x_1) - L| \ge \epsilon$$

$$\exists x_2 \text{ s.t. } 0 < |x_2 - a| < \frac{1}{2} \text{ and } |f(x_2) - L| \ge \epsilon$$

$$\vdots$$

$$\exists x_n \text{ s.t. } 0 < |x_n - a| < \frac{1}{n} \text{ and } |f(x_n) - L| \ge \epsilon$$

for all $n \in \mathbb{Z}_{pos}$. Thus (x_n) converges to a with $x_n \neq a \forall n$ and $\lim_{n \to \infty} f(x_n) \neq L$.

The nice thing about this result is that it allows us to port most of our results about limits of sequences to analogous results about limits of functions. For example, this provides the most expedient approach to proving algebra of limits results for functions.

2. Continuity

Last time, Pedro gave us the following intuition: a function f is continuous if you can draw f without lifting your writing implement from the writing surface.

Defining continuity rigorously turns out to be quite tricky. Let's try to define continuity in terms of what it's *not*: what would it mean for f to be *discontinuous*?

We guessed that a function might be discontinuous if

(1) the function is not defined somewhere

- (2) the function has a "hole"
- (3) the function has a "jump"



How do we define this rigorously?

Definition (Lily). f is discontinuous at a if and only if $\lim_{x\to a} f(x) \neq f(a)$

Does this hold in our three scenarios above? In the second and third cases, it works. But in the first case, f(a) is not defined, so the statement in our definition is vacuously true and vacuously false. We fix this by requiring f(a) to be defined:

Definition (Informal). f is continuous at a if and only if f(a) exists and $\lim_{x \to a} f(x) = f(a)$.

Isn't it a little strange that we're defining continuity at a single point? How could this relate to Pedro's idea of continuity about being able to draw the function without lifting your pen? In fact, this is *distinct* from Pedro's informal definition. Formally:

Definition. Given $f: X \to \mathbb{R}$, $a \in X$, we say f is continuous at a if and only if $\forall \epsilon > 0 \exists \delta > 0$ such that

$$|x-a| < \delta$$
 and $x \in X \implies |f(x) - f(a)| < \epsilon$

If f is continuous at all points in X, we say f is continuous on X.

Remark. Unlike our limit definition, we do <u>not</u> require a to be an accumulation point of X. Why is this? With limits there is too much ambiguity; we observed that the limit at an isolated point could be literally anything. With continuity, on the other hand, it's a binary choice: we can either define the function to be continuous or not, and we choose to say it is.

Example 1.

$$f:[0,1]\cup\{2\}$$
$$x\mapsto x^2$$



Is f continuous at 1? Yes, because for all the inputs you plug in that are close to 1 (from the left, since the function is not defined to the right of 1), you get an output that is close to 1.

Is f continuous at 2? Yes. Given $\epsilon > 0$, let $\delta = \frac{1}{2}$. Observe that

$$|x-2| < \frac{1}{2}$$
 and $x \in [0,1] \cup \{2\} \implies |x^2-4| < \epsilon$.

Even though the limit of f(x) is not defined at x = 2, the function is continuous at 2.

Example 2. The function $f : \mathbb{Z} \to \mathbb{R}$ given by $x \mapsto 3$ is continuous everywhere.

Example 3. The same function defined on \mathbb{Q} is continuous.

Example 4. The function $f : \mathbb{Q} \to \mathbb{R}$ defined by $x \mapsto \frac{1}{x^2-2}$ is continuous on \mathbb{Q} because the only potentially problematic points are $\pm \sqrt{2}$, but these are not in our domain.

3. INTERMEDIATE VALUE THEOREM

We ended class by stating a famous theorem about continuity:

Theorem 3.1 (Intermediate Value Theorem). Given $f : X \to \mathbb{R}$ continuous on X. Then $\forall a, b \text{ such that } [a, b] \in X, \forall y \text{ between (inclusive) } f(a) \text{ and } f(b), \exists c \in [a, b] \text{ s.t. } f(c) = y.$

This, in fact, is Pedro's original statement about continuity, but it is not equivalent to continuity. It is a *consequence* of continuity, but it turns out the converse is false. There are functions satisfying the conclusion of IVT that are not continuous. These are called Darboux functions. Next lecture, we will prove IVT and see examples of Darboux functions.