REAL ANALYSIS: LECTURE 23

DECEMBER 4TH, 2023

1. PRELIMINARIES

Last time, we discussed continuity. The main idea is that f is continuous at a iff $\lim_{x\to a} f(x) = f(a)$. Writing this out using the $\epsilon - \delta$ definition of limits, we get the following:

Definition (Continuity- Da Formal Version)**.** Given $f : X \to \mathbb{R}$ with $\emptyset \neq X \subseteq \mathbb{R}$ and $a \in X$, we say f is *continuous* at a iff $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$(|x-a| < \delta) \wedge (x \in X) \implies |f(x) - f(a)| < \epsilon.$$

Further, we say f is continuous on $S \subseteq \mathbb{R}$ iff f is continuous at a $\forall a \in S$, and that f is continuous iff f is continuous on X (i.e. its domain).

Remark. A consequence of this formulation is that f is continuous at isolated points in X. Why? Well, $a \in X$ is isolated iff you can zoom in closely enough such that you can only see a. Let's say for clarity that if you are within δ of a then you can't see anything else. Then, no matter what ϵ is, notice that

$$(|x-a| < \delta) \wedge (x \in X) \implies |f(x) - f(a)| < \epsilon$$

is vacuously true, meaning by our formal definition f is continuous at a.

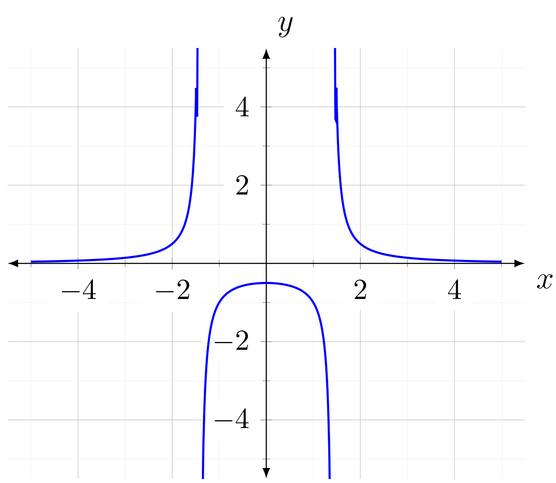
Let's do some examples:

Example 1. Say $X \subseteq \mathbb{R}$ is a nonempty finite set. Let $f : X \to \mathbb{R}$. Is f continuous? Matt says yes! X consists only of isolated points, and we know that f is continuous at any points, which means f is continuous.

Example 2. Define

$$f : \mathbb{Q} \to \mathbb{R}$$
$$x \mapsto \frac{1}{x^2 - 2}.$$

Let's draw this.



Is this continuous? Well visually no. But where are the jumps? Only at $\pm\sqrt{2}$, i.e. only at irrational values. Thus, on $\mathbb{Q}$ it does actually behave nicely, meaning it is continuous.

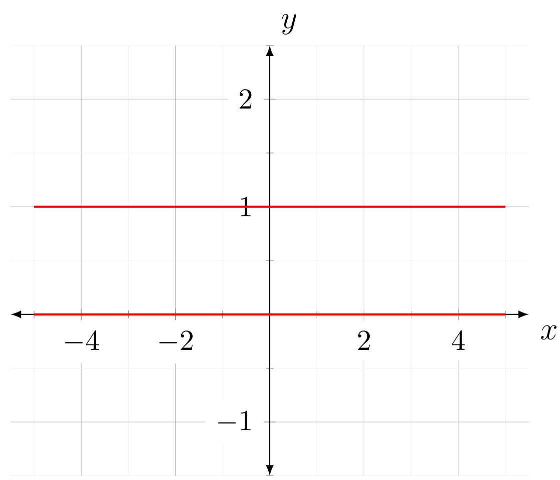
Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

Remark. Noah had a good point. $\mathbb{Q}$ has *holes*, so is this function also just a bunch of isolated points? Well, not really. No matter how close you look, there are still rational numbers, meaning these points aren't quite isolated.

Example 3. This is known as the *terrid function*. Define

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Here's what this looks like:

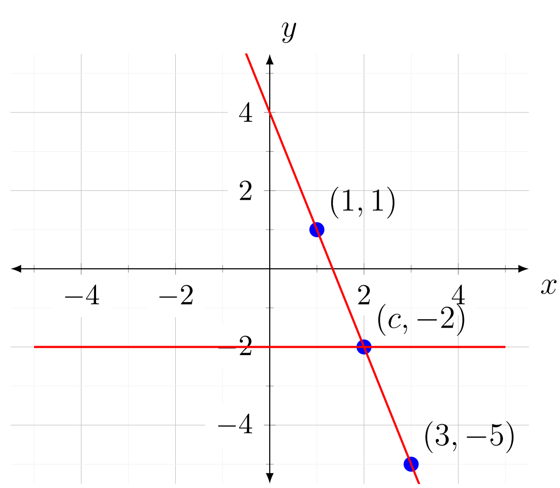


This is actually a well defined function, though it's difficult to tell. Is this continuous? Turns out it's *not continuous* at any $x \in \mathbb{R}$. Intuitively, this is true because no matter how close you zoom in on any particular point, there are rational and irrational numbers nearby, meaning limits won't work well.

If you remember way back not that long ago, we said that continuity was roughly like drawing and not lifting up your pencil. But this is completely non-obvious from our $\epsilon - \delta$ definition! It turns out it's more analogous to the following theorem:

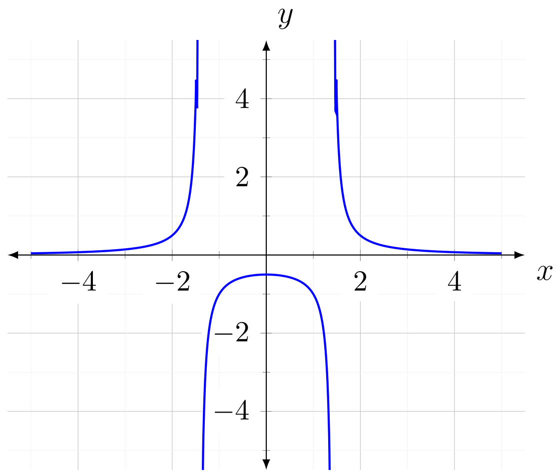
Theorem 1 (Intermediate Value Theorem)**.** *Given $f : X \to \mathbb{R}$ continuous. For any $[a, b] \subseteq X$ and for any y in between $f(a)$ and $f(b)$, $\exists c \in [a, b]$ s.t. $f(c) = y$.*

Remark.



Here $a = 1, b = 3$ and $f(a) = 1, f(b) = -5, y = -2$. The IVT states that there must be some $1 < c < 3$ s.t. $f(c) = -2$.

Remark. Edith has a great point here. What about our function $f : \mathbb{Q} \to \mathbb{R}$ from before?



If you take $a = -5, b = 0$, note $f(-5) = 1/23, f(0) = -1/2$, yet there's no $c \in [-5, 0]$ such that $f(c) = 0$ for example. Miles pointed out that there's a hidden assumption within IVT: the domain must include an interval for the theorem to hold. Within the statement of the theorem we say $\forall [a, b] \subseteq X$. But, $[-5, 0] \not\subseteq \mathbb{Q}$, since for example $-\sqrt{2} \in [-5, 0] \setminus \mathbb{Q}$.

Is the converse true? Here's the question:

Question. We call a function $f : X \to \mathbb{R}$ *Darboux* if it satisfies the conclusion of the IVT. Namely, $f : X \to \mathbb{R}$ is *Darboux* iff $\forall [a, b] \subseteq X$, $\forall y$ between $f(a), f(b)$, there exists $c \in [a, b]$ s.t. $f(c) = y$.

So, must Darboux functions be continuous?

Jenna says no. Here's a specific example:

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

I'd plot this one on Desmos. It's basically a wave where things get wild at 0. In particular, it's not continuous at 0, though one can show it is in fact Darboux. Another crazy one is something known as Conway's base 13 function. Ok, let's explore a weird consequence of the IVT:

Claim. There exists two points on the surface of earth that are antipodal (on opposite ends of a diameter) at which the temperatures are identical.

Proof. Here's a meta-analytic proof. Let $T(x)$ be the temperature of $x \in \text{EARTH}$. Pick any antipodal points p, q. Rotate diameter connecting p and q by $\theta \in [0, \pi]$, i.e. until p and q switch places. Consider the function

$$f(\theta) := T(p_\theta) - T(q_\theta),$$

where x_θ is the point x rotated by θ. WLOG suppose $f(0) > 0$. Note $f(\pi) = -f(0) < 0$. Thus, by IVT, $\exists c \in [0, \pi]$ s.t.

$$f(c) = 0 \iff (T(p_c) = T(q_c)).$$

$\square$

Here's another application:

Theorem 2 (Wobbly Table Theorem)**.** *Imagine you have a table with* 4 *legs that are all the same length. Now, the table could be wobbly, since the ground might be a bit wild. Then there exists a rotation of the table such that the table is stable.*

Remark. Yeah... this kinda relies on a few assumptions that might not be true. For example, if the ground is not smooth it won't work.

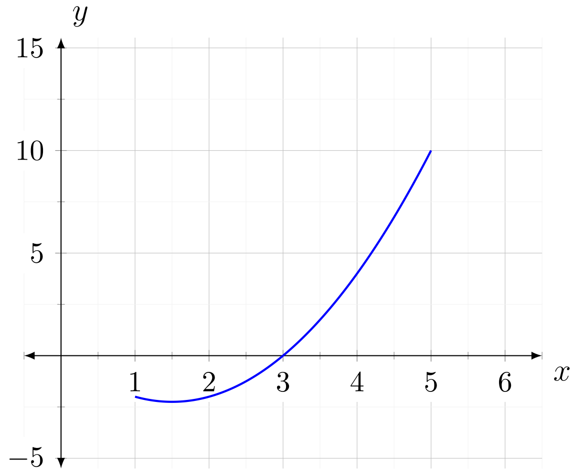
Ok, so we looked at two very useful practical applications. Let's actually prove the IVT. It turns out it suffices to prove a special case:

Theorem 3 (Bolzano's Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and*

$$f(a) < 0 < f(b),$$

then there exists $c \in [a, b]$ s.t. $f(c) = 0$.

Scratchwork 1. *Here's a picture:*



Ok, the function is negative at $x = 1$, and then it stays that way for a while before $f(x_0) = 0$. So, let's just look at all the x values where $f(x) < 0$ and take the supremum of that. As a first guess we might do something like:

$$\mathcal{A} := \{x \in [a, b] \ : \ f(x) < 0\}.$$

Miles mentions that if $f(x) = 0$ at multiple x then this set will actually give the largest x s.t. $f(x) = 0$. It turns out this adds some technical difficulties, so let's just revamp the set:

$$\mathcal{A} := \{x \in [a, b] \ : f(x_0) < 0 \ \forall x_0 \in [a, x]\}.$$

We're then going to take $c := \sup \mathcal{A}$, and we'll show $f(c) \neq 0$. In particular, we'll show $f(c) \not> 0$ and $f(c) \not< 0$. Why? Well, say $f(c) > 0$. There's intuitively room ϵ such that $f(c - \epsilon) > 0$, which is a contradiction.

Proof. Here's the main idea of the proof. Let

$$\mathcal{A} := \{x \in [a, b] \ : f(x_0) < 0 \ \forall x_0 \in [a, x]\}.$$

Notice $a \in \mathcal{A}$ and $b \geq x_0 \ \forall x_0 \in \mathcal{A}$. Let $c := \sup \mathcal{A}$, which exists by (A13). Suppose, for the sake of contradiction, that $f(c) > 0$. Since f continuous at c, $\exists \delta > 0$ s.t.

$$\begin{aligned}
|x - c| < \delta &\implies |f(x) - f(c) < \frac{f(c)}{2} \\
&\implies f(x) - f(c) > -\frac{f(c)}{2} \\
&\implies f(x) > f(c) > 0 \\
&\implies f\left(c - \frac{\delta}{2}\right) > 0.
\end{aligned}$$

Thus if $f(c) > 0, \exists d < c$ s.t. $f(d) > 0$, which contradicts c being the *least* upper bound. We'll finish the rest of this next time. $\square$