REAL ANALYSIS: LECTURE 24

DECEMBER 7TH, 2023

1. Real Analysis- The Finale

Golly to the gee. It's the last class of Real Analysis (don't cry too much or else you can't read this summary). Just a mere couple of months ago we were all so naive; we didn't even know the real numbers are the only complete ordered field up to isomorphism! Look at us now. It only takes us like 10 minutes to prove $f(x) = 12$ is continuous! Ok, but honestly we learned a lot this semester. Real numbers: super weird. Infinity: also super weird. Limits: a lot more annoying to define than you might think. ϵ: your new favorite Greek letter. Log trivia: Thursday. Real Analysis: cool, but difficult. Ok, enough of this. Let's do some math.

2. IVT

Last time, we were proving the IVT. Let's try to remember what the IVT is.

Idea 1. *Here's Alex's idea. For any* $f : [a, b] \to \mathbb{R}$ *continuous,* $\forall y$ *between* $f(a), f(b), \exists c \in [a, b]$ *s.t.* $f(c) = y$.

This is correct, but IVT is actually stronger. Noah notes it's not that $f : [a, b] \to \mathbb{R}$. Rather, for *every* closed interval that lives in the domain, the above property is true. Namely,

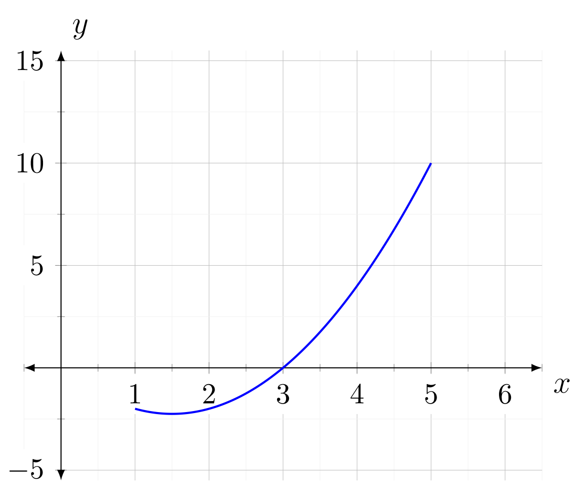
Theorem 1 (IVT). *Given* $f : X \to \mathbb{R}$ *continuous. For any* $[a, b] \subseteq X$ *and for any* y *in between* $f(a)$ *and* $f(b)$, $\exists c \in [a, b]$ *s.t.* $f(c) = y$.

Our goal is to deduce IVT from a weaker theorem, called Bolzano's Theorem. Here's the statement again:

Theorem 2 (Bolzano's Theorem). *If* $f : X \to \mathbb{R}$ *is continuous on* $[a, b] \subseteq X$ *and*

$$f(a) < 0 < f(b),$$

then there exists $c \in [a, b]$ *s.t.* $f(c) = 0$.



Here's a picture. Call the function above f and note $f(1) < 0 < f(6)$. Bolzano's Theorem states that there must be some $x \in [1, 6]$ s.t. $f(c) = 0$. Note there may be many c's! Ok, let's prove Bolzano's Theorem by beginning with the following lemma:

Lemma 1. *If* $f : [a, b] \to \mathbb{R}$ *continuous at* $c \in [a, b]$ *and* $f(c) > 0$, *then* $\exists \delta > 0$ *s.t.* $f > 0$ *on all of* $(c - \delta, c + \delta)$.

Notes on a lecture by Leo Goldmakher; written by Justin Cheigh.

Idea 2. *Here's the intuition. If you zoom in on $x = c$, the stuff nearby must be close to $f(c) > 0$, which means the stuff nearby should be > 0 as well. More specifically we can look at some ϵ wiggle-room. Let's prove this.*

Proof. Since f continuous at c, $\exists \delta > 0$ s.t. $|x - c| < \delta \implies |f(x) - f(c)| < \frac{f(c)}{2}$ (here $\frac{f(c)}{2}$ is our choice of ϵ). In particular,

$$\begin{aligned} x \in (c - \delta, c + \delta) &\implies f(x) - f(c) > -\frac{f(c)}{2} \\ &\implies f(x) > \frac{f(c)}{2} > 0. \end{aligned}$$

Thus $f > 0$ everywhere on the interval $(c - \delta, c + \delta)$ as claimed. □

Remark. Notice Lemma 1 works if you replace all the $f(c) > 0$ with $f(c) < 0$. Why? Well just take a function g where you want to make the claim with $f(c) < 0$. Multiplying everything by -1 gives you a new function $f = -g$, which meets the conditions of Lemma 1 and gives us the desired result.

Let's use Lemma 1 to prove Bolzano's Theorem. Here's the high level idea

Scratchwork 1. *Last time we decided if we defined*

$$c := \sup \underbrace{\{x \in [a, b] \ : \ f < 0 \text{ on all of } [a, x]\}}_{=:\mathcal{A}}$$

then it makes sense that $f(c) = 0$. We can then use trichotomy to prove it. So, what's the problem with $f(c) > 0$. Well, on the one hand we can go slightly left of the supremum and find some d s.t. $f(d) < 0$ (since $d \in \mathcal{A}$). On the other hand, we can use the above lemma and claim that d is close enough to c that both should be positive; namely $f(d) > 0$, which gets us the contradiction. At a high level, this contradiction arises by the fact that there's room to the left (c is not the least upper bound). On the case where $f(c) < 0$, we should be able to move over to the right and still find stuff that lives in $\mathcal{A}$, i.e. c is not an upper bound!

Proof. Define c as above; we proved last class that $c \in \mathbb{R}$. To show $f(c) = 0$, it suffices (by trichotomy) to show $f(c) \not> 0$ and $f(c) \not< 0$.

Case 1. *Suppose, for the sake of contradiction, $f(c) > 0$. By Lemma 1, $\exists \delta > 0$ s.t. $f(x) > 0 \, \forall x \in (c - \delta, c + \delta)$.*

Remark. Here's a subtly flawed approach. Using the above statement we can deduce that

$$f\left(c - \frac{\delta}{2}\right) > 0,$$

which makes it seem like we're done. However, we don't know that $c - \frac{\delta}{2} \in \mathcal{A}$!

Ok, let's keep going. Since $c = \sup \mathcal{A}$, we know $(c - \delta, c]$ contains some $d \in \mathcal{A}$. Notice $c \neq d$, since $f(c) > 0$, which means $c \notin \mathcal{A}$ (it doesn't meet the set condition!). But $d \in \mathcal{A}$ means $f(d) < 0$, yet on the other hand $d \in (c - \delta, c + \delta)$ implies $f(d) > 0$. Contradiction!

Let's move to the other case:

Case 2. *Suppose, for the sake of contradiction, $f(c) < 0$. By Lemma 1 (actually by our remark afterwards), $\exists \delta > 0$ s.t. $f(x) < 0 \, \forall x \in (c - \delta, c + \delta)$. In particular, $f\left(c + \frac{\delta}{2}\right) < 0$. But also $\exists d \in (c - \delta, c]$ s.t. $d \in \mathcal{A}$. Thus, $f < 0$ on $[a, d]$ and $f < 0$ on $\left[d, c + \frac{\delta}{2}\right]$ since $d \in (c - \delta, c + \delta)$. Thus, $c + \frac{\delta}{2} \in \mathcal{A}$, which means c isn't an upper bound on $\mathcal{A}$.*

Together these cases tell us $f(c) = 0$, and we're done! □

Now let's prove IVT, using Bolzano's Theorem.

Proof. Given $f : X \to \mathbb{R}$ continuous. Pick $[a, b] \subseteq X$. If $f(a) < y < f(b)$, apply Bolzano to $g(x) := f(x) - y$ (here $g(a) < 0 < g(b)$ so Bolzano applies), which gives us $c \in [a, b]$ s.t. $g(c) = 0 \implies f(c) = 0 + y = y$, as desired. On the other hand, if $f(a) > y > f(b)$, apply Bolzano to $h(x) = -f(x) + y$, which gives us $c \in [a, b]$ s.t. $h(c) = 0 \implies f(c) = y - h(c) = y$. □

3. Series

Ok, 15 minutes left. Time to move to a new topic: series. Apparently, the following is true:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \cdots = 1.$$

What does this actually mean? How can you add up an infinite amount of things. Here's what this really means. Let

$$S_N := \sum_{n=1}^{N} \frac{1}{2^n}.$$

Then (S_N) is a sequence of what we call *partial sums*, where

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{4} \\ S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\ &\vdots \end{aligned}$$

For any particular choice of N, we know what S_N means—it's just a finite sum. The equation

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

is merely *notation* that represents

$$\lim_{N\to\infty} S_N = 1.$$

More generally, we write

$$\sum_{n=1}^{\infty} a_n = L$$

iff the sequences of partial sums

$$S_N := \sum_{n=1}^{N} a_n$$

converges to L.

Example 1. In Lecture 17, we proved

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by showing the Cauchy Criterion with telescoping series. This series is known as $\zeta(2)$ (Riemann Zeta Function). It turns out this series converges to $\pi^2/6$.

Example 2. Also in Lecture 17, we proved

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. This is known as the *Harmonic Series*.

Example 3. Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

This is known as the *Alternating Harmonic Series*. Let's show a sketch of why this diverges.

Proof. For each $N \in \mathbb{Z}_{\text{pos}}$, let

$$S_N := \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$$

be the partial sum of the series. Observe that

$$S_{2N} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2N-1} - \frac{1}{2N}\right)$$

is a sum of positive terms, and in particular the subsequence S_{2N} is monotonically increasing. If we could show that (S_{2N}) is bounded above, the MCT would imply that it converges! Let's see:

$$S_{2N} = S_{2N-1} - \frac{1}{2N} < S_{2N-1}$$

and

$$S_{2N-1} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2N-2} - \frac{1}{2N-1}\right) < 1.$$

Thus $S_{2N} < S_{2N-1} < 1$, which tells us by the MCT that S_{2N} converges. In particular, there exists some real number L such that $S_{2N} \to L$. In particular, $\forall \epsilon > 0$ we have

$$\begin{aligned} |S_{2N} - L| &< \frac{\epsilon}{2} \qquad \text{for all large } N \\ |S_{2N-1} - L| &= |S_{2N} + \frac{1}{2N} - L| \\ &\leq |S_{2N} - L| + \frac{1}{2N} \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon \qquad \text{for all large } N. \end{aligned}$$

Thus for any large $k \in \mathbb{Z}_{\text{pos}}$, we see that $|S_k - L| < \epsilon$, whence (S_N) converges. We've proved that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. (It turns out it converges to $\log 2$, amazingly!) □

Congrats! Real Analysis: Complete!