Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 350 : REAL ANALYSIS

Midterm Exam – Solutions

NAME (PRINT): ______________________________
Last First Nickname

NAME (PRINT): ______________________________
Last First Nickname

NAME (PRINT): ______________________________
Last First Nickname

INSTRUCTIONS: This exam will take place in two stages.

STAGE I (50 MINUTES). This phase of the exam is a typical in-class exam: you will work individually, with no communication between you and any other person, and without any external aids or reference materials. At the end of the 50 minutes, please **remain seated** until all exams have been collected.

STAGE II (25 MINUTES). Find the other members of your assigned group and sit together (and, as much as possible, apart from the other groups); you are allowed to find some space outside of the exam room, if you like. During this phase, your group will be expected to collaborate on the exam questions, and then to write up a single set of solutions as a group. In other words, your group must arrive at a consensus on a solution to each question; you may not submit multiple solutions to any question. Make sure to write all your names (first and last) on your submission. **Each person in the group must write down the solution to at least one problem** (where indicated, record who wrote up each problem).

Your solutions from Stage I will count as 80% of your exam score; your solutions to Stage II will count 20%. However, *under no circumstance will the collaborative grade lower your individual exam score.*

You are allowed to use any result from class or the book without proving it (unless I specify otherwise in the question).

Please sign below the honor code below prior to starting the exam. As soon as you've signed and the proctor has announced the start of the exam, you may begin.

Best of luck!!

I understand that any breach of academic integrity is a violation of the Honor Code. By signing below, I pledge to abide by the Code.

SIGNATURE:______________________________

M.1 (10 points) Does the sequence $a_n := 2 + \frac{(-1)^n}{n}$ converge or diverge? Justify your answer with a formal ϵ-style proof. (*You may only use the definition of the limit for this problem. In particular, do not use the Cauchy criterion or the algebra of limits.*)

Claim. $a_n \to 2$.

Proof. Given $\epsilon > 0$. For any $n > \frac{1}{\epsilon}$ we have

$$|a_n - 2| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} < \epsilon.$$

□

M.2 Let $S := \{0, 1, 2, 3, 4, 5\}$ with addition (denoted $+$) and multiplication (denoted $\cdot$) defined by the following tables:

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	3	4	5	1
3	0	3	4	5	1	2
4	0	4	5	1	2	3
5	0	5	1	2	3	4

For each of the following assertions, circle True or False and write a short informal (meta-analytic) justification for your answer.

(a) (3 points) S satisfies (A8) (multiplication is commutative). **(True)** False

Short Justification:

The multiplication table is symmetric across the southeast diagonal.

(b) (3 points) S satisfies (A10) (existence of multiplicative inverses). **(True)** False

Short Justification:

From the multiplication table, 1 is clearly a multiplicative identity. Every nonzero row of the multiplication table contains a 1, which means every nonzero element of S has a multiplicative inverse.

(c) (3 points) S satisfies (A11) (distributivity). True **(False)**

Short Justification:

Observe that

$$3 \cdot (2 + 3) = 3 \cdot 5 = 2$$
$$3 \cdot 2 + 3 \cdot 3 = 4 + 5 = 3$$

so the distributive property fails.

M.3 (16 points) Prove that $\sqrt[3]{2} \in \mathbb{R}$. In other words, prove that there exists $\alpha \in \mathbb{R}$ such that $\alpha^3 = 2$. [*You may not use Theorem 7.5 from the book.*]

Let $\mathcal{A} := \{x \in \mathbb{R} : x^3 < 2\}$, and set $\alpha := \sup \mathcal{A}$. Note that $\alpha \in \mathbb{R}$ by (A13), since $\mathcal{A}$ is nonempty ($1 \in \mathcal{A}$) and bounded above (2 is an upper bound on $\mathcal{A}$: if $2 < x \in \mathcal{A}$ then $2^3 < x^3 < 2$, a contradiction). Further note that $\alpha \geq 1$, since $1 \in \mathcal{A}$ and α is an upper bound on $\mathcal{A}$.

Claim. $\alpha^3 = 2$.

Proof. We prove this by trichotomy.

- Can $\alpha^3 > 2$?

  Suppose $\alpha^3 > 2$. Then $\alpha^3 - 2 > 0$, so by Archimedean Property there exists a positive integer $n > \frac{3\alpha^2+1}{\alpha^3-2}$. We deduce

$$\begin{aligned}\left(\alpha - \tfrac{1}{n}\right)^3 &= \alpha^3 - \tfrac{3}{n}\alpha^2 + \tfrac{3}{n^2}\alpha - \tfrac{1}{n^3} \\ &\geq \alpha^3 - \tfrac{3}{n}\alpha^2 \qquad\qquad - \tfrac{1}{n} \qquad\qquad (\text{since } \tfrac{1}{n^3} \leq \tfrac{1}{n}) \\ &= \alpha^3 - \tfrac{3\alpha^2+1}{n} \\ &> \alpha^3 - (\alpha^3 - 2) = 2.\end{aligned}$$

  But this would mean that $\alpha - \frac{1}{n}$ is an upper bound on $\mathcal{A}$, which contradicts that α is the *least* upper bound on $\mathcal{A}$! Thus $\alpha^3 \not> 2$.

- Can $\alpha^3 < 2$?

  Suppose $\alpha^3 < 2$. Then $2 - \alpha^3 > 0$, so by Archimedean Property there exists a positive integer $n > \frac{3\alpha^2+3\alpha+1}{2-\alpha^3}$. We deduce

$$\begin{aligned}\left(\alpha + \tfrac{1}{n}\right)^3 &= \alpha^3 + \tfrac{3}{n}\alpha^2 + \tfrac{3}{n^2}\alpha + \tfrac{1}{n^3} \\ &\leq \alpha^3 + \tfrac{3}{n}\alpha^2 + \tfrac{3}{n}\alpha + \tfrac{1}{n} \\ &= \alpha^3 + \tfrac{3\alpha^2+3\alpha+1}{n} \\ &< \alpha^3 + (2 - \alpha^3) = 2.\end{aligned}$$

  But this would mean that $\alpha + \frac{1}{n} \in \mathcal{A}$, contradicting that α is an upper bound on $\mathcal{A}$! Thus $\alpha^3 \not< 2$.

Since $\alpha^3 \not< 2$ and $\alpha^3 \not> 2$, trichotomy implies $\alpha^3 = 2$, as claimed. □

M.4 (10 points) Suppose $\mathcal{A}$ is a nonempty set. Prove that there does not exist any surjection $\mathcal{A} \twoheadrightarrow \mathcal{P}(\mathcal{A})$. [*Here* $\mathcal{P}(\mathcal{A})$ *denotes the power set of* $\mathcal{A}$.]

Pick any $f : \mathcal{A} \to \mathcal{P}(\mathcal{A})$, and set

$$\mathcal{B} := \{x \in \mathcal{A} : x \notin f(x)\} \subseteq \mathcal{A}.$$

By definition, $\mathcal{B} \in \mathcal{P}(\mathcal{A})$.

Claim. $\mathcal{B} \notin \operatorname{im} f$.

Proof. Suppose $\mathcal{B} \in \operatorname{im} f$. Then $\exists a \in \mathcal{A}$ such that $f(a) = \mathcal{B}$. Then

$$a \in \mathcal{B} \quad \iff \quad a \notin f(a) \quad \iff \quad a \notin \mathcal{B}.$$

But this is preposterous! □

We've thus found an element of $\mathcal{P}(\mathcal{A})$ that's not in the image of f, showing that f cannot be a surjection.

M.5 (5 points) Suppose $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a function such that

(i) $f(\alpha)$ is a nonempty open interval for each $\alpha \in \mathbb{R}$, and

(ii) for any $\alpha, \beta \in \mathbb{R}$, either $f(\alpha) = f(\beta)$ or $f(\alpha) \cap f(\beta) = \varnothing$.

Prove that f isn't injective. [*Here* $\mathcal{P}(\mathbb{R})$ *denotes the power set of* $\mathbb{R}$.]

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, condition (i) implies that for any $x \in \mathbb{R}$, the open interval $f(x)$ contains some rational number $q_x \in \mathbb{Q}$. Now consider the function

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{Q} \\ x &\longmapsto q_x \end{aligned}$$

This map cannot be injective, since $\mathbb{R}$ is uncountable while $\mathbb{Q}$ is countable; thus there must exist real numbers $\alpha \neq \beta$ with $q_\alpha = q_\beta$. In particular, $q_\alpha \in f(\alpha) \cap f(\beta)$, so condition (ii) implies $f(\alpha) = f(\beta)$. We conclude that f cannot be injective.