Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 350 : REAL ANALYSIS

Problem Set 2 – due Thursday, September 21st

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 4pm on Friday.

(0) Read Chapters 2-3 (pages 4-12).

(1) Book problems 2.1, 2.3, 3.3, 3.5, 3.6, and 3.8 [*In 2.1, "range" means "image" and* f^{-1} *means* $f^{-1}(\text{im } f)$.]

(2) Given a function $f : A \to B$. Prove the following statements without referring to Theorem 2.6 of the book (since the relevant parts of that theorem aren't proved there).

(a) If $X, Y \subseteq B$, then $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.

(b) If $X, Y \subseteq A$, then $f(X \cup Y) = f(X) \cup f(Y)$.

(3) Suppose a set S consists of exactly three (distinct) elements and satisfies (A1)–(A4). Must there exist $x \in S$ such that $x + x \neq 0$? Either prove such an x must exist, or give an example of a set S in which there's no such x.

(4) (Meta-analytic) For each of the following sets and operations, identify all the axioms out of (A1)-(A5) that don't hold. Whenever an axiom fails to be satisfied, give an example illustrating the failure.

(i) The set $\mathbb{Z} \setminus \{0\}$ where + means multiplication (e.g. $3 + 5 := 15$).

(ii) The set $\mathcal{P}\big(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\big)$ (the power set) where + means intersection (i.e. $a + b := a \cap b$).

(iii) The set of all positive integers, where + is defined by $a + b := \gcd(a, b)$. (Recall that given two positive integers a and b, the *greatest common divisor* of a and b, denoted $\gcd(a, b)$, is the largest positive integer dividing both a and b.)

(iv) The set of all positive integers, where + is defined by $a + b := \text{lcm}(a, b)$. (Recall that given two positive integers a and b, the *least common multiple* of a and b, denoted $\text{lcm}(a, b)$, is the smallest positive integer that is a multiple of both a and b.)

(v) The set of all non-negative integers (i.e. $\{0, 1, 2, \ldots\}$), where + is defined by $a + b := |a - b|$.

(5) (Meta-analytic) A person is walking through a moving train from the back to the front. How quickly is the person moving relative to the ground? According to Einstein, the right way to add two velocities v_1, v_2 pointing in the same direction is by the rule

$$v_1 \mathbin{@} v_2 := \frac{v_1 + v_2}{1 + v_1 v_2 / c^2},$$

where c denotes the speed of light. Prove that the interval $I = (-c, c)$ under the binary operation @ satisfies axioms (A1)-(A5). (In particular, in this model nothing can move faster than the speed of light!)

(6) In Theorem 3.4 of the book, it's shown that $x \cdot 0 = 0$ for all $x \in \mathbb{R}$. Here's an alternative proof:

$$\begin{aligned} x \cdot 0 &= x \cdot (-1+1) \qquad \text{by (A5)} \\ &= x \cdot (-1) + x \cdot 1 \qquad \text{by (A11)} \\ &= -x + x \qquad \text{by (A9)} \\ &= 0 \qquad \text{by (A5).} \qquad \square \end{aligned}$$

There's a very good reason the book didn't give this proof. What is it? (Your answer can be very short, so long as it is compelling.)