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MATH 350 : REAL ANALYSIS

Problem Set 5 – due Friday, October 13th

INSTRUCTIONS:

This assignment is due by 4pm on Friday, October 13th, to be left in the mailbox outside my office door; no late penalty will be assigned, but assignments will not be accepted after 4pm on Friday.

(0) Read Chapter 7 (pages 20-24).

(1) Prove that for any $x \in \mathbb{R}$ there exists $N \in \mathbb{Z}$ and $\alpha \in [0, 1)$ such that $x = N + \alpha$, and that this choice of N and α are uniquely determined by x. (Recall from class that N is called the *floor* of x, denoted $\lfloor x \rfloor$, and α is called the *fractional part* of x, denoted $\{\alpha\}$. For example, $\lfloor \pi \rfloor = 3$ and $\{\pi\} = 0.1415926\ldots$)

[NOTE. *In class we proved the above statement for $x \geq 1$. You may use that result without reproving it!*]

(2) In class, we proved that $\sqrt{2} \in \mathbb{R}$. Earlier, we gave a meta-analytic proof that $\sqrt{2} \notin \mathbb{Q}$. The goal of this problem is to give an analytic proof that $\sqrt{2}$ is irrational due to John Conway (1937-2020). Here and throughout, set $\mathbb{Q}_{\text{pos}} := \{\frac{a}{b} : a, b \in \mathbb{Z}_{\text{pos}}\}$; we call the elements of this set the *positive rational numbers*. Let

$$\mathcal{A} := \{m \in \mathbb{Z}_{\text{pos}} : m\sqrt{2} \in \mathbb{Z}_{\text{pos}}\}.$$

(a) Give the simplest colloquial (i.e. meta-analytic) description of the set $\mathcal{A}$ you can come up with. [*Hint: try to use the word 'denominator'.*]

(b) Prove that if $a, b \in \mathbb{Z}_{\text{pos}}$, then there exists $c \in \mathbb{Z}_{\text{pos}} \cup \{0\}$ such that $0 \leq c < b$ and

$$\left\{\frac{a}{b}\right\} = \frac{c}{b}.$$

[*In the first line, $\{0\}$ denotes the set with the single element 0; in the displayed equation, $\left\{\frac{a}{b}\right\}$ denotes the fractional part of $\frac{a}{b}$.*]

(c) Suppose $n \in \mathcal{A}$. Prove that there must exist $k \in \mathbb{Z}_{\text{pos}}$ such that $\sqrt{2} = \frac{k}{n} = \frac{2n}{k}$.

(d) Keeping the notation as above, prove that $\exists k', n' \in \mathbb{Z}$ such that $0 < k' < k$, $0 < n' < n$, and $\frac{n'}{n} = \frac{k'}{k}$.

(e) Keeping the notation as above, prove that $n' \in \mathcal{A}$.

(f) Prove that $\mathcal{A} = \emptyset$. Why does this imply that $\sqrt{2} \notin \mathbb{Q}_{\text{pos}}$?

(3) (Meta-analytic) Let $\mathbb{Z}[x]$ denote the set of all polynomials with integer coefficients, and consider the set

$$\mathcal{F} := \left\{\frac{f(x)}{g(x)} : f, g \in \mathbb{Z}[x] \text{ s.t. } g \text{ isn't the constant 0 function}\right\}.$$

We'll define $h \in \mathcal{F}$ to be *positive* iff $h(x) > 0$ for all large $x \in \mathbb{R}$. Prove that $\mathcal{F}$ is an ordered field, i.e. satisfies (A1)-(A12), but that it fails to satisfy the Archimedean Property. This demonstrates that the Archimedean Property cannot be deduced from (A1)-(A12) alone!

(4) (Meta-analytic) Textbook problem 7.10

(5) (Meta-analytic) Use induction to prove that $2^{2^n} - 1$ has at least n distinct prime factors.

[*Hint. If two integers a and b are both multiples of n, then so is $a - b$.*]