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MATH 350 : REAL ANALYSIS

Problem Set 6 – due Thursday, October 24th

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 4pm on Friday.

Please type up Problems (2), (3), and (5) in \mathbb{IAT}_{EX} .

(0) Read Chapters 8-9 (pages 26-32).

Some convenient notation:

- The notation $f: A \hookrightarrow B$ means that $f: A \to B$ is injective.
- The notation $f: A \twoheadrightarrow B$ means that $f: A \to B$ is surjective.
- The notation $f: A \hookrightarrow B$ means that $f: A \to B$ is bijective.
- The notation $A \hookrightarrow B$ (read: 'A injects into B') means there exists an injection $f : A \hookrightarrow B$.
- The notation $A \twoheadrightarrow B$ (read: 'A surjects onto B') means there exists a surjection $f: A \twoheadrightarrow B$.
- (1) Prove that if $f : A \hookrightarrow B$, then $A \approx f(A)$.
- (2) Find an explicit bijection $(0,1] \hookrightarrow (0,1)$. Your function is allowed to be defined piecewise, so long as you explicitly state where each element of (0,1] gets sent.
- (3) The goal of this exercise is to provide a method of testing whether a given set is countable without producing a bijection with \mathbb{Z}_{pos} .
 - (a) Prove that $\mathbb{Z}_{pos} \hookrightarrow A$ for any infinite set A. (This answers Armie's question in class: there's no infinite set strictly smaller than \mathbb{Z}_{pos} .)
 - (b) Suppose $A \hookrightarrow \mathbb{Z}_{pos}$. Prove that A must be countable.
 - (c) Prove that if $A \hookrightarrow \mathbb{Z}_{pos}$ and $\mathbb{Z}_{pos} \hookrightarrow A$ then $A \approx \mathbb{Z}_{pos}$.
- (4) In class we sketched a meta-analytic argument for \mathbb{Q}_{pos} being countable. Here we give a rigorous proof.
 - (a) Prove that if $A \hookrightarrow B$ and $B \hookrightarrow C$ then $A \hookrightarrow C$.
 - (b) Prove that any positive integer can be written in the form $2^k n$, where $k \in \mathbb{Z}_{pos} \cup \{0\}$ and n is a positive odd integer.
 - (c) Use part (b) to give an explicit bijection $\mathbb{Z}_{pos} \times \mathbb{Z}_{pos} \hookrightarrow \mathbb{Z}_{pos}$. Prove your map is a bijection.
 - (d) Construct an explicit injection $\mathbb{Q}_{pos} \hookrightarrow \mathbb{Z}_{pos} \times \mathbb{Z}_{pos}$. [Caution: make sure your map is a function!]
 - (e) Combine parts (a), (c), and (d) to give a short, rigorous proof that \mathbb{Q}_{pos} is countable.

- (5) Given a set A, let \mathcal{F} denote the set of all functions $f : A \to \{0, 1\}$. Prove that $\mathcal{F} \approx \mathcal{P}(A)$, and use this to explain why some use the notation 2^A rather than $\mathcal{P}(A)$.
- (6) Textbook problems: 8.2, 9.4, 9.3, 9.10 (the last problem is meta-analytic)

Challenge Problems (Not for submission, but I'm happy to discuss them!)

- (7*). If $A \approx [0,1]$, prove that $A \times A \approx A$. [This is true for any infinite A, but it's much harder to prove for A strictly larger than [0,1].]
- (8*). Textbook problem 7.9. [Note that by "one-to-one" the book means the function f is a bijection.]