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MATH 350 : REAL ANALYSIS

Problem Set 6 – due Thursday, October 26th

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 4pm on Friday.

(0) Read Chapters 8-9 (pages 26-32).

(1) Suppose $f : A \hookrightarrow B$. Prove that $A \approx f(A)$.

(2) Find an explicit bijection $(-1, 1) \hookrightarrow\!\!\!\!\rightarrow \mathbb{R}$. [*Meta-analytic, but don't use functions we haven't defined in class.*]

(3) Find an explicit bijection $(0, 1] \hookrightarrow\!\!\!\!\rightarrow (0, 1)$. Your function is allowed to be defined piecewise, so long as you explicitly state where each element of $(0, 1]$ gets sent. [*Hint: Where should you send 1?*]

(4) The goal of this exercise is to prove a simple case of Cantor-Schröder-Bernstein (see part (c)).

(a) Prove that of all infinite sets, $\mathbb{Z}_{\text{pos}}$ has the smallest size, i.e. that $\mathbb{Z}_{\text{pos}} \hookrightarrow A$ for any infinite set A.

(b) Suppose $A \hookrightarrow \mathbb{Z}_{\text{pos}}$. Without using Cantor-Schröder-Bernstein, prove that A must be countable.

(c) Prove (without Cantor-Schröder-Bernstein) that if $A \hookrightarrow \mathbb{Z}_{\text{pos}}$ and $\mathbb{Z}_{\text{pos}} \hookrightarrow A$ then $A \approx \mathbb{Z}_{\text{pos}}$.

(5) In class we sketched an argument for $\mathbb{Q}_{\text{pos}}$ being countable. Here we give a rigorous proof of this.

(a) Prove that if $A \hookrightarrow B$ and $B \hookrightarrow C$ then $A \hookrightarrow C$. [*Colloquially: if B is at least as large as A, and C is at least as large as B, then C is at least as large as A.*]

(b) Prove that any positive integer can be written in the form $2^k n$, where $k \in \mathbb{Z}_{\text{pos}} \cup \{0\}$ and n is a positive odd integer.

(c) Use part (b) to give an explicit bijection $\mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}} \hookrightarrow\!\!\!\!\rightarrow \mathbb{Z}_{\text{pos}}$. Prove your map is a bijection.

(d) Construct an explicit injection $\mathbb{Q}_{\text{pos}} \hookrightarrow \mathbb{Z}_{\text{pos}} \times \mathbb{Z}_{\text{pos}}$.

(e) Combine parts (a), (c), and (d) to give a short, rigorous proof that $\mathbb{Q}_{\text{pos}}$ is countable.

(6) Given a set A, let $\mathcal{F}$ denote the set of all functions $f : A \to \{0, 1\}$. Prove that $\mathcal{F} \approx \mathcal{P}(A)$, and use this to explain why some use the notation 2^A rather than $\mathcal{P}(A)$.

(7) Textbook problems: **8.2, 9.4, 9.3, 9.10** (the last problem is meta-analytic)

(8) Textbook problem **7.9**. [Note that by "one-to-one" the book means the function f is a bijection.]

Challenge Problem (Not for submission, but I'm happy to discuss it with you.)

(9*). If $A \approx [0, 1]$, prove that $A \times A \approx A$. [This is true for any infinite A, but it's much harder to prove for A strictly larger than $[0, 1]$.]