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MATH 350 : REAL ANALYSIS

Problem Set 7 – due Thursday, November 2nd

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 4pm on Friday.

(0) Read Chapters 10-12 (pages 34-44).

(1) Checking some fundamentals...

(a) Is $\infty \in \mathbb{R}$? Carefully formulate what properties you'd like such a number to have, and then prove that it is or isn't an element of $\mathbb{R}$.

(b) Suppose $|x| \leq \epsilon$ for every $\epsilon > 0$. Prove that $x = 0$.

(c) Use (b) to prove that $0.\overline{9} = 1$, where $0.\overline{9}$ denotes the number $0.9999\cdots$ written in decimal notation.

(d) Prove that there does not exist a smallest positive real number.

(2) Textbook problems **10.3, 10.5, 10.7, 10.12**

(3) Consider the sequence

$$a_n := \begin{cases} 1 & \text{if } n = 2^k \text{ for some } k \in \mathbb{Z}_{\text{pos}} \\ \frac{1}{n} & \text{otherwise,} \end{cases}$$

so the sequence begins $1, 1, \frac{1}{3}, 1, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, 1, \frac{1}{9}, \ldots$ Does (a_n) converge? Justify your answer with a proof.

(4) Suppose (a_n) and (b_n) are convergent sequences and $a_n < b_n$ for all $n \in \mathbb{Z}_{\text{pos}}$. Does it follow that $\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$? Either prove this, or provide a counterexample.

(5) We call a sequence (x_n) *bounded* iff the set $\{x_n : n \in \mathbb{Z}_{\text{pos}}\}$ is bounded. Show (by example) that it's possible to have a bounded sequence (a_n) and a convergent sequence (b_n) such that both $(a_n + b_n)$ and $(a_n b_n)$ diverge.

(6) Given a sequence (a_n), set $b_n := a_{2n} - a_n$.

(a) Suppose (a_n) converges. Must (b_n) converge? Justify your answer with a proof based on the definition of limit (i.e. without using theorems about limits).

(b) Give an example of (a_n) that diverges such that (b_n) diverges.

(c) Give an example of (a_n) that diverges such that (b_n) converges.

(7) Consider the sequence $a_n := n + \frac{(-1)^n}{n}$. Does (a_n) converge? Justify your answer with a proof.

BONUS This is just for fun—not to be submitted, but I'd be delighted to hear what you come up with!

Consider the sequence (a_n) defined by

$$a_1 = a_2 = 1, \qquad a_{2n} := a_n, \qquad \text{and} \qquad a_{2n+1} := a_n + a_{n+1}.$$

Thus the sequence begins $1, 1, 2, 1, 3, 2, 3, 1, \ldots$ This is called *Stern's diatomic sequence*, and it has a number of remarkable properties. Here are a couple of these.

(a) Prove that the map $\mathbb{Z}_{\text{pos}} \to \mathbb{Q}_{\text{pos}}$ defined by $n \mapsto \frac{a_n}{a_{n+1}}$ is a bijection. (This gives an explicit enumeration of the rationals, which we didn't do in class.)

(b) Recall Pascal's triangle:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Prove that the sum along the diagonals of Pascal's triangle produces the sequence of Fibonacci numbers. For example, the diagonal that's circled sums to 21.

(c) Draw Pascal's triangle (mod 2), i.e. change each even entry to 0 and each odd entry to 1:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 0 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 1 & & 1 & & & & \\
 & & & 1 & & 0 & & 0 & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & \\
 & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

Explain why this looks like the Sierpinski gasket.

(d) Prove that the sum along the diagonals of Pascal's triangle (mod 2) produces Stern's diatomic sequence a_n.