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MATH 350 : REAL ANALYSIS

Problem Set 8 – due Thursday, November 16th

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (incurring a small penalty, as described in the course syllabus). Assignments will not be accepted after 4pm on Friday.

(0) Read Chapters 13–16 (pages 45-53).

(1) Determine (with proof) $\lim_{n\to\infty} \sqrt{n+3} - \sqrt{n}$.

(2) Compute $\lim_{n\to\infty} \sqrt[n]{2}$. Do not use Theorem 16.4. [*Hint. Start by proving that* $\left(1+\frac{1}{n}\right)^n \geq 2$.]

(3) Compute $\lim_{n\to\infty} \sqrt[n]{1+\frac{n}{n+1}}$. [*Hint. Squeeze theorem!*]

(4) Suppose $a_1 > 1$, and let $a_{n+1} = 2 - 1/a_n$ for each positive integer n. Prove that (a_n) converges, and (rigorously) find its limit.

(5) Suppose $0 < a_1 < b_1$ and for each positive integer n define

$$a_{n+1} := \sqrt{a_n b_n} \qquad\qquad b_{n+1} := \frac{a_n + b_n}{2}$$

(a) Prove that both sequences (a_n) and (b_n) converge.

(b) Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(6) Consider the following two sequences:

$$(a_n) \quad : \quad \sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

$$(b_n) \quad : \quad \sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$$

More formally, we can define $a_1 = b_1 = \sqrt{2}$ and for each positive integer n set

$$a_{n+1} := \sqrt{2a_n} \qquad \text{and} \qquad b_{n+1} := \sqrt{2+b_n}.$$

Finally, we define a third sequence:

$$c_n := \frac{2^n}{b_1 b_2 \cdots b_n}$$

(a) Prove that (a_n) converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)

(b) Prove that (b_n) converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)

(c) Prove that (c_n) converges, and make a conjecture about what it converges to. (You don't have to prove your conjecture, but you should try to explain where it comes from.)