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MATH 350 : REAL ANALYSIS

Problem Set 9 – due Thursday(ish!), November 30th

INSTRUCTIONS:

You should aim to submit this assignment to me in person at the start of Thursday's class; if you cannot make it to class, email me by 11am on Thursday and we can discuss alternative ways to submit your assignment. Late assignments can be left in the mailbox outside my office until 4pm on Friday (no late penalty this week!). Assignments will not be accepted after 4pm on Friday.

(0) Read Chapters 18 and 19. (We won't cover chapter 17, but you might wish to skim it for general education.)

(1) In class, we outlined a proof of the Cauchy criterion which is similar to that given in Chapter 19 of the book. The biggest difference between our proof and the book's is the approach to the Bolzano-Weierstrass theorem; our proof was based on Miles' and Ben's insight that any sequence has a monotone subsequence, while the book's proof is a clever binary search algorithm. The goal of this problem is to make our approach rigorous. Given a sequence (a_n), we call a_k a *peak* iff $a_k \geq a_m$ for all $m \geq k$.

(a) Suppose (a_n) has infinitely many peaks. Prove that (a_n) has a monotone subsequence.

(b) Suppose (a_n) has finitely many peaks. Prove that (a_n) has a monotone subsequence.

(c) Deduce the Bolzano-Weierstrass theorem from the previous parts.

(2) For each of the following metrics on $\mathbb{R}^2$, draw a picture the open ball $\mathcal{B}_3\big((2,0)\big)$. No proofs necessary.

(a) The chessboard metric $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(b) The British Rail metric

$$d(x,y) := \begin{cases} |x| + |y| & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

(Here $|x|$ denotes the Euclidean distance from x to the origin.)

(c) The discrete metric $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$

(3) Suppose (X,d) is a metric space and $\mathcal{A} \subseteq X$. We say $p \in X$ is an *interior point* of $\mathcal{A}$ iff $\exists r > 0$ such that $\mathcal{B}_r(p) \subseteq \mathcal{A}$, and that $p \in X$ is a *limit point* of $\mathcal{A}$ iff there exists a sequence (a_n) of points in $\mathcal{A}$ such that $\lim\limits_{n\to\infty} a_n = p$. (As always, $\mathcal{B}_r(p)$ denotes the ball of radius r around p.)

(a) Prove that $\mathcal{A}$ is open iff every point of $\mathcal{A}$ is an interior point of $\mathcal{A}$. (In class we defined: $\mathcal{A}$ is open iff $\partial\mathcal{A} \cap \mathcal{A} = \varnothing$.)

(b) Prove that $\mathcal{A}$ is closed iff every limit point of $\mathcal{A}$ is in $\mathcal{A}$. (In class we defined: $\mathcal{A}$ is closed iff $\partial\mathcal{A} \subseteq \mathcal{A}$.)

(4) Suppose (X,d) is a metric space. Prove that $\mathcal{B}_r(p)$ is open for any $p \in X$ and any $r > 0$.

(5) Decide (with proof or counterexample) whether each of the following is a metric space.

(a) The set $\{a, b, c, d\}$ with the distance between any two of a, b, c being 2 and the distance between d and any one of a, b, c being 1.

(b) $\mathbb{R}^\infty := \{(a_n) : (a_n) \text{ is a sequence of real numbers}\}$, with respect to $d(x, y) := \max\{|x_n - y_n|\}$.

(c) $\mathcal{F} := \{A \subseteq \mathbb{Z} : A \text{ is finite}\}$, with respect to $d(X, Y) := \log \frac{|X-Y|}{\sqrt{|X|}\sqrt{|Y|}}$. Here $|S|$ denotes the size of S and $X - Y := \{x - y : x \in X, y \in Y\}$.

(6) Exploring metrics on $\mathbb{R}^2$.

(a) Prove that the Euclidean metric on $\mathbb{R}^2$ is, in fact, a metric.

(b) Suppose $\mathcal{O}$ is a subset of $\mathbb{R}^2$ that's open with respect to the Euclidean metric. Must it also be open with respect to the taxicab metric?

(c) The Euclidean and taxicab metrics on $\mathbb{R}^2$ both have the form

$$d_p(x, y) := \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}$$

(d_1 is the taxicab metric, d_2 is the Euclidean metric). It turns out that d_p is a metric for every real number $p \geq 1$. (Don't worry about proving it here, although it is a fun challenge to think about when you have some spare time.) Can you describe any of the other metrics on $\mathbb{R}^2$ that we've encountered (chessboard, British Rail, and discrete) in terms of d_p? No formal proofs necessary, but give a bit of justification for your answer.

(7) Given a metric space (X, d) where X has at least 2 elements. Prove that there exists a metric on X that's not a scalar multiple of d or of the discrete metric.

(8) Given (X, d) a metric space and $\mathcal{A} \subseteq X$. Prove that $\mathcal{A}$ is closed iff $\mathcal{A}^c$ is open.

(*) (*Optional challenge problem—won't be graded*) Let $M_{n\times n}$ denote the space of all $n \times n$ matrices with real entries. Prove that $d(x, y) := \operatorname{rank}(x - y)$ is a metric on $M_{n\times n}$.